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A POLYNOMIAL TIME BRANCH AND BOUND ALGORITHM FOR THE SINGLE ITEM ECONOMIC LOT SIZING PROBLEM WITH ALL UNITS DISCOUNT AND RESALE

S.H. Mirmohammadi[*, †], Sh. Shadrokh and K. Eshghi
Department of Industrial Engineering, Isfahan University of Technology, Isfahan, Iran

ABSTRACT

The purpose of this paper is to present a polynomial time algorithm which determines the lot sizes for purchase component in Material Requirement Planning (MRP) environments with deterministic time-phased demand with zero lead time. In this model, backlog is not permitted, the unit purchasing price is based on the all-units discount system and resale of the excess units is possible at the ordering time. The properties of an optimal order policy are argued and on the basis of them, a branch and bound algorithm is presented to construct an optimal sequence of order policies. In the proposed B&B algorithm, some useful fathoming rules have been proven to make the algorithm very efficient. By defining a rooted tree graph, it has been shown that the worst-case time complexity function of the presented algorithm is polynomial. Finally, some test problems which are randomly generated in various environments are solved to show the efficiency of the algorithm.

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1. INTRODUCTION

In a large scale project with nonrenewable resources or in a manufacturing firm, the task of replenishing components at the right time, price and quantities has an essential effect on the total cost of the project. When the demand rate changes over time and replenishments are

[*]Corresponding author: S. Hamid Mirmohammadi, Department of Industrial Engineering, Isfahan University of Technology, Isfahan, Iran
[†]E-mail address: h_mirmohammadi@cc.iut.ac.ir (H. Mirmohammadi)

made periodically, the problem of ordering a single product over a finite horizon and satisfying the demands without backlogging is known as the general dynamic lot size problem. This problem has been studied by many researchers. In particular, when all-units quantity discounts are available, the problem is called Quantity Discount Problem (QDP). Benton and Park [1] have separated the literature on solving QDP into two categories: exact methods and heuristic methods. The exact methods find an optimal order policy that minimizes the total inventory costs. Chung et al. [2] have developed an optimal dynamic algorithm for the QDP and they have proved an essential property of the optimal order policy. Federgruen and Lee [3] have proposed a dynamic programming algorithm for the QDP with only one discount level in purchasing. They claimed that their algorithm is an optimal algorithm of $O(N^3)$ where N is the number of periods in the planning horizon, but Xu and Lu [4] by presenting some special counterexamples have shown that their algorithm fails to find the optimal solution in some cases. Mirmohammadi et al. [5] have presented an optimal algorithm based on the branch and bound approach for the QDP which is extremely more efficient than Chung et al. algorithm [2], especially for large scale problems. Chan et al. [6] have shown that the QDP becomes NP-hard if the purchasing cost function of the amount ordered satisfies the three following properties:

(i) it is a nondecreasing function of the amount ordered,
(ii) the purchasing cost per unit is nonincreasing in the amount ordered and
(iii) it either varies from period to period or the number of breakpoints is not bounded

The QDP is also applicable for production planning problems as Hoesel and Wagelmans [7] have developed an algorithm that solves the constant capacitated economic lot-sizing problem with concave production costs and linear holding costs in $O(N^3)$ time. Their greedy algorithm is based on the standard dynamic programming approach which is based on structural properties of the optimal sub plans to arrive at a more efficient implementation. When the number of items which should be acquired is more than one, the QDP changes into the total quantity discount (TQD). Goossens et al. [8] have proved that TQD is *NP*-hard and also there exists no polynomial-time approximation algorithm with a constant ratio for this problem (unless $P = NP$). When all-units discount are available from vendors, under some circumstances, buying a sufficiently large quantity to qualify for a certain discount and then disposing the excess units (with a positive or negative cost per unit) to save on inventory holding cost, leads to economic polices [9]. Sethi [9] has considered the simple lot size mode with quantity discount and allowing the possibility of disposal at some finite cost in the environment with constant demand rate. The case with negative disposal cost per unit is considered as the resale in the literature which is modeled by Sohan and Hwang [10]. With respect to the running time of the algorithm. Sohan and Hwang [10] have observed that their algorithm is in $O(N^3(\frac{d_1^N}{q})^2)$ where d_1^N is the cumulative demand of all periods and q is the discount level. It is obvious that the time complexity function of their algorithm depends on the demand rate. For example if the item to be acquired has a demand pattern with constant average, the time complexity function of their algorithm becomes $O(N^5)$. It can be observed in real-life situations that the demand for essential commodities such as petrol, diesel and for sophisticated items such as electronic goods, computer spare parts, etc. increases gradually with time. So, their demand pattern can be represented appropriately by a linear increasing function of time, Giria et al. [11]. In these cases, their algorithm becomes $O(N^7)$.

The organization of this paper is as follows. In section 2, the dynamic quantity discount lot size model with resale is discussed. The assumptions and the properties of an optimal order policy for the single level discount case are explained in section 3. In section 4, an optimal branch and bound algorithm and a numerical example are presented. The worst-case time complexity function of the presented algorithm is studied in section 5 by defining a rooted tree graph. Section 6 presents an experimental design for evaluating our algorithm in some varied environments. Concluding remarks are made in section 7.

2. THE DYNAMIC QUANTITY DISCOUNT LOT SIZE MODEL WITH RESALE

Consider a planning horizon of N periods. It is assumed that there is a known positive demand for an item at each period that should be met by some orders through these periods, and backlog is not allowed. We assume that any ordering can occur only at the beginning of each period with constant ordering cost of A. The ordered items arrive immediately to satisfy the demand of that period. At this time, it is also possible to resell some units of the arrived order at a constant price.

Now, for period t, $t = 1, 2, ..., N$, let

d_t = amount demanded

I_t = amount of inventory at the end of period, $I_0 = I_N = 0$

h_t = holding cost per unit of inventory carried from period j to period $j+1$

$d_t^j = \sum_{i=t}^{j} d_i$, the cumulative demands from period t to period j, $t \le j \le N$.

the constant parameters are

U = unit net purchasing price

A = ordering cost

C = unit resale price

There are two decision variables for period t, $t = 1, 2, ..., N$, as stated below:

x_t = amount ordered

r_t = amount resold.

It is assumed that there is one discount rate a, $0 < a \le 1$, which is associate with the price break point (discount level) D, $D > 0$. The unit purchase price of x_t is $(1-a)U$ if $x_t \ge D$, otherwise it is U. A reasonable assumption is that $C < (1-a)U$. The problem is to find x_t and r_t, $t = 1, 2, ..., N$, such that all demands are met at the minimum total cost. The model can be formulated as follows:

$$Min\ \ Z = \sum_{t=1}^{N} \left[A.\,l\,(x_t) + p(x_t) + h_t.I_t - C.r_t \right]$$

s.t. $I_t = I_{t-1} + x_t - d_t - r_t$, $t = 1,2,...,N$

$I_t \geq 0$, $t = 1, 2,..., N$

$I_0 = I_N = 0,$

x_t , $r_t \geq 0$ and *Integer*, $t = 1,2,...,N$

where $l(x_t) = \begin{cases} 1 & x_t > 0 \\ 0 & otherwise \end{cases}$ and

$$p(x_t) = \begin{cases} (1-a)Ux_t & x_t \geq D \\ Ux_t & otherwise \end{cases} \tag{1}$$

The purchasing cost function which is depicted in Figure 1 is a time independent and piecewise linear function.

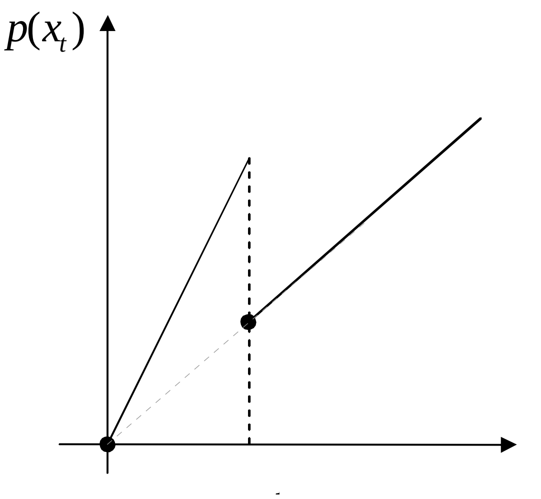

Figure 1. The purchasing cost function

3. THE PROPERTIES OF THE OPTIMAL ORDER POLICY

The properties of an optimal order policy are presented in this section. These properties let us to develop the optimal branch and bound algorithm in the next section. Property 1 and Property 2 have proven by Sohan and Hwang [10] but Property 3 and Property 4 are proved here. At first, we need to define the following terms.

Fraction period: Let period t , $t = 1, 2,..., N$, be a "fraction period" whenever $x_t \neq D$ and $x_t > 0$.

Resale period: Let period t , $t = 1, 2,..., N$, be a "resale period" whenever $r_t > 0$.

Sub plan: Let in an optimal order policy $I_u = I_v = 0$, $0 \leq u < v \leq N$, and $I_t > 0$ for $u < t < v$. $SP_{uv} = \{u+1, u+2,..., v\}$ is defined as a "sub plan".

Property 1. There exists an optimal order policy such that each one of its sub plans has the following properties:

1-1- It includes at most one fraction period.

1-2- It includes at most one resale period.

1-3- It does not include both fraction period and resale period.

Property 2. If any resale occurs in SP_{uv}, then $u+1$ is the resale period.

Property 3. Let $f_{u,v}$ be a fraction period in SP_{uv} and $l_{u,v} = \max_{u<t\le v}\{t \mid x_t > 0\}$. Then $f_{u,v} = l_{u,v}$.

Proof. Assume on the contrary that $f_{u,v} < l_{u,v}$. By Property 1-1, we have $x_{l_{u,v}} = D$. Since $f_{u,v}$ is a fraction period, then $x_{f_{u,v}} \ne D$. Let $P_{f_{u,v}}$ and $P_{l_{u,v}}$ be the unit purchasing price for the lots $x_{f_{u,v}}$ and $x_{l_{u,v}}$, respectively. Decrease $x_{f_{u,v}}$ by 1 and also increase $x_{l_{u,v}}$ by 1 (it is possible since $I_t > 0$ for $f_{u,v} \le t < l_{u,v}$). The net decrease in the objective value of the optimal order policy is $P_{f_{u,v}} - P_{l_{u,v}} + \sum_{i=f_{u,v}}^{l_{u,v}-1} h_i$. If it is positive, the optimality is contradicted and the proof is completed, otherwise, we have

$$P_{l_{u,v}} - P_{f_{u,v}} \ge \sum_{i=f_{u,v}}^{l_{u,v}-1} h_i \tag{2}$$

In this case move the ordering of period $l_{u,v}$ to period $f_{u,v}$ in other words, increase $x_{f_{u,v}}$ by $x_{l_{u,v}}$ and omit the ordering at $l_{u,v}$. The net decrease in the objective value of the optimal order policy is at least $A + x_{l_{u,v}}(P_{l_{u,v}} - P_{f_{u,v}} - \sum_{i=f_{u,v}}^{l_{u,v}-1} h_i)$ which is positive by (2). This contradicts the optimality, and the proof is completed. □

Lemma 1. Let $N(u,t)$, $0 \le u < t \le v \le N$, be the total number of ordering occurred from $u+1$ to t when SP_{uv} has a resale period and $r(u,v)$ be the resale amount at $u+1$ in this case (i.e. $r(u,v) = r_{u+1}$). We have

$$N(u,t) = \left\lceil \frac{d_{u+1}^{t} + r(u,v)}{D} \right\rceil \tag{3}$$

$$r(u,v) = \left\lceil \frac{d_{u+1}^{v}}{D} \right\rceil D - d_{u+1}^{v} \tag{4}$$

Proof. If SP_{uv} has a resale period, then $x_t \in \{0, D\}$, $t = u+1, u+2, ..., v$, by Property 1-3. Therefore, to cover the demands of periods $u+1, u+2, ..., t$, it is necessary that

$N(u,t) \geq \left\lceil \frac{d_{u+1}^{t} + r(u,v)}{D} \right\rceil$. If $N(u,t) > \left\lceil \frac{d_{u+1}^{t} + r(u,v)}{D} \right\rceil$, then $I_t > D$, but this incurs an additional inventory holding cost and contradicts the optimality. Now, Let n be the total number of ordering occurred through SP_{uv}. Since there is no fraction period in SP_{uv} by Property 1-3, we have $r(u,v) = nD - d_u^v$. Since $I_u = I_v = 0$, we have $n \geq \left\lceil \frac{d_{u+1}^{v}}{D} \right\rceil$. If $n > \left\lceil \frac{d_{u+1}^{v}}{D} \right\rceil$, then $r(u,v) > D$ which contradicts the optimality by the assumption of $C < (1-a)U$ and the proof is complete. □

The following property can be derived from previous properties.

Property 4. In the optimal order policy each sub plan like SP_{uv}, $0 \leq u < v \leq N$, has one of the following forms:

4-1- $r_{u+1} = 0$, $x_l = d_l^v - I_{l-1}$ and $x_t \in \{0, D\}$ for $u+1 \leq t \leq v$, $t \neq l_{u,v}$, where $l_{u,v} = \max_{u<j\leq v}\{j \mid x_j > 0\}$.

4-2- $r_{u+1} = r(u,v)$ and $x_t \in \{0, D\}$ for $u+1 \leq t \leq v$ where $r(u,v)$ is obtained by (4).

Proof. By Property 1 SP_{uv} has at most one fraction period; therefore, $x_t \in \{0, D\}$ for $u+1 \leq t \leq v$, $t \neq l_{u,v}$. If $r_{u+1} = 0$, then $x_{l_{u,v}} = d_{l_{u,v}}^{v} - I_{l_{u,v}-1}$ since $l_{u,v}$ is the last period in which an ordering occurred and $I_u = I_v = 0$. Consider the case $r_{u+1} > 0$. By Lemma 1 $r_{u+1} = r(u,v)$ and by Property 1-3 $x_t \in \{0, D\}$ therefore, the proof is complete. □

Corollary 1. If SP_{uv} has a resale period , i.e. $r_{u+1} > 0$, the set of orders occurred through SP_{uv} is $\Phi(u,v)$ where

$$\Phi(u,v) = [(D, r(u,v)), (D,0), (D,0), ..., (D,0)] \tag{5}$$

Such that $|\Phi(u,v)| = N(u,v)$ and the i^{th}, $i = 1, 2, ..., N(u,v)$, ordering of $\Phi(u,v)$ occurred at period $t(i)$, $u+1 \leq t(i) \leq v$ where $t(1) = u+1$ and

$$t(i) = \max_{u+2 \leq j \leq v}\{j \mid (i-1)D - r(u,v) \geq d_{u+1}^{j-1}\} \tag{6}$$

Proof. The number of ordering occurred through SP_{uv}, the amount ordered and the amount resole are determined by Property 4-2. Since $I_u = 0$, it is obvious that $t(1) = u+1$. Let $t(i) = j$. The remaining inventory at the beginning of period j is $(i-1)D - r(u,v) - d_{u+1}^{j-1}$. Since backlog is not permitted $(i-1)D - r(u,v) \geq d_{u+1}^{j-1}$. On the other hand to reach the minimum inventory holding cost, an ordering must occur only when the remaining inventory is not sufficient enough to cover the forward period. This means that

$t(i) = \max_{u+2 \le j \le v} \{ j \mid (i-1)D - r(u,v) \ge d_{u+1}^{j-1} \}$ and the proof is complete. □

To determine whether resale is economical or not when the order size differs from D and to reach an upper bound for the resale amount an order size break point is defined. This break point is denoted by $D_{0.5}$ and is calculated by Eq. (7):

$$D_{0.5} = \frac{(1-a)U - C}{U - C} D \tag{7}$$

If quantity x, $D_{0.5} \le x \le D$, is to be acquired, then it is more economical to purchase D units and resell $D - x$. The next lemma obtains an upper bound for the resale amount.

Lemma 2. For the resale period of SP_{uv}, $u+1$, we have $0 \le r_{u+1} \le D - D_{0.5}$.

Proof. If SP_{uv} has no resale period, Property 2 implies $r_{u+1} = 0$ and the proof is complete; otherwise, assume that $r_{u+1} > D - D_{0.5}$. Since $x_{u+1} = D$ by Property 1-3, if $D - r_{u+1}$ units are purchased directly without any resale, a lower purchasing cost will be incurred, and this is impossible because SP_{uv} is a sub plan of the optimal order policy. Hence, the proof is complete. □

Sometimes a sub plan like SP_{uv} may have only one ordering at the beginning of $u+1$ without any resale. By comparing the objective values, it is possible to distinguish this case from the case in which SP_{uv} has a resale period. Let $Z_{so}(u,v)$ and $Z_r(u,v)$ be the objective values in these cases respectively. We have

$$Z_{so}(u,v) = A + d_{u+1}^{v} p(d_{u+1}^{v}) + \sum_{t=u+1}^{v-1} h_t (d_{u+1}^{v} - d_{u+1}^{t}) \tag{8}$$

$$Z_r(u,v) = N(u,v).(A + D(1-a)U) + \sum_{t=u+1}^{v-1} h_t (N(u,t).D - r(u,v) - d_{u+1}^{t}) - C.r(u,v) \tag{9}$$

where $p(d_{u+1}^{v})$, $N(u,j)$ and $r(u,v)$ are obtained by (1), (3) and (4), respectively. By the following theorem we are able to construct the optimal order policy which has the above properties.

Theorem 1. Let (x_t, r_t) be the optimal order policy in period t, $t = 1, 2, ..., N$.

1- If $I_{t-1} \ge d_t$, then $(x_t, r_t) = (0,0)$.

2- If $0 < I_{t-1} < d_t$, then $(x_t, r_t) \in S_t^0$ where $S_t^0 = \{(d_t^v - I_{t-1}, 0) \mid v = t, t+1, ..., N\} \cup \{(D,0)\}$

3- If $I_{t-1} = 0$, then $(x_t, r_t) \in S_t^1$ such that $S_t^1 = \{(x_t^v, r_t^v) \mid v = t, t+1, ..., N\} \cup \{(D,0)\}$ where

$$(x_t^v, r_t^v) = \begin{cases} (d_t^v, 0) & \textbf{\textit{if}}\ r(t-1,v) > D - D_{0.5}\ \textbf{\textit{or}}\ Z_r(t-1,v) \ge Z_{so}(t-1,v) \\ (D, r(t-1,v)) & \textbf{\textit{otherwise}} \end{cases}$$

and $r(t-1,v)$ is obtained by (4).

Proof. When $I_{t-1} \geq d_t$ due to a lower inventory holding cost, the optimality imposes $(x_t, r_t) = (0,0)$. A sub plan, say SP_{uv}, exists such that $t \in SP_{uv}$ and $1 \leq u+1 \leq t \leq v \leq N$. When $0 < I_{t-1} < d_j$ by the definition of SP_{uv}, we have $t \geq u+2$; therefore, $r_t = 0$ by Property 2. On the other hand, by Property 4-1 we have $x_t = d_t^v - I_{t-1}$ when t is the fraction period of SP_{uv}, otherwise $x_t = D$, hence $(x_t, r_t) \in \{(d_t^v - I_{t-1}, 0), (D,0)\}$ for this case. Now, let $I_{t-1} = 0$ therefore, $t = u+1$ which means that a resale may occur at t. If $r(t-1, v) > D - D_{0.5}$ or $Z_r(t-1, v) \geq Z_{so}(t-1, v)$, then SP_{uv} has no resale period and by Property 4-1 $x_t = d_t^v$ (if t is the fraction period) or $x_t = D$ (if t is not the fraction period); otherwise, $(x_t, r_t) = (D, r(t-1, v))$ (if SP_{uv} has a resale period) or $(x_t, r_t) = (D,0)$ (if SP_{uv} has no resale period). Hence, for this case, $(x_t, r_t) \in \{(d_t^v, 0), (D,0)\}$ if $r(t-1, v) > D - D_{0.5}$, or $Z_r(t-1, v) \geq Z_{so}(t-1, v)$ otherwise, $(x_t, r_t) \in \{(D, r(t-1, v)), (D,0)\}$. By $v = t, t+1, ..., N$ the proof is complete. □

By Theorem 1 the possible alternatives for (x_t, r_t), $t = 1, 2, ..., N$, are reduced to a finite set of order policies when I_{t-1} is known. In other words, if $I_{t-1} \geq d_t$ then $(x_t, r_t) = (0,0)$, otherwise $(x_t, r_t) \in S_t$ where

$$S_t = \begin{cases} S_t^0 & if \quad I_{t-1} > 0 \\ \\ S_t^1 & if \quad I_{t-1} = 0 \end{cases} \tag{10}$$

Note that in the case that the condition of resale are held (i.e. $r(t-1, v) \leq D - D_{0.5}$ or $Z_r(t-1, v) < Z_{so}(t-1, v)$) all order policies in $\{t, t+1, ..., v\}$ are determined by Corollary 1 and $v+1$ is the next reordering point. Since $|S_t| \leq N - t + 2$, there are at most $N - t + 2$ different candidate for the order policy in period t, $t = 1, 2, ..., N$, and therefore, the optimal order policy of the problem can be found among at most $\prod_{j=1}^{N} (N - j + 2) = (N+1)!$ different alternative policies. A branch and bound algorithm is presented in the next section to enumerate these policies implicitly.

4. A BRANCH AND BOUND ALGORITHM

Now, we are able to construct the sequence of orders in the optimal order policy for the problem. Starting from period 1 and assuming $I_0 = 0$, we calculate the set of alternatives for (x_1, r_1), or S_1. Each order policy in S_1, like (x_1, r_1), covers the demand up to a period, say t, $t \geq 2$, with a definite inventory I_{t-1} which are obtained by Eqs. (11) and (12), respectively.

$$t = \max_{2 \le j \le N+1} \{ j \mid x_1 - r_1 \ge d_1^{j-1} \} \tag{11}$$

$$I_{t-1} = x_1 - r_1 - d_1^{t-1} \tag{12}$$

Period t is called a "reordering point" because the remaining inventory (I_{t-1}) is not enough to cover the demand of any forward periods. Each reordering point has a corresponding partial policy which is defined as the ordered set of those order policies that cover the demand up to the reordering point. The corresponding partial policy of t is denoted by P^t where $P^t = [(x_1, r_1)]$ in this case. Now, having t and I_{t-1}, S_t is obtained and for each feasible order policy $(x_t, r_t) \in S_t$, the corresponding reordering point with its inventory are calculated. This process is repeated till at least one of the fathoming rules, derived in this section, occurs. In order to follow this process more easily, we implement it in the form of a search tree which is formed of nodes and edges. A node in the search tree contains the information of a reordering point and its corresponding partial policy. This information is summarized in four elements. In details, each node is denoted by $[t, I_{t-1}, Z, P^t]$ where:

t : The reordering point; the period in which the node is ended to, $t = 1, \mathbf{2,...,} N$

I_{t-1} : The remaining inventory of the node; obviously $I_{t-1} < d_t$

P^t : The partial policy of the node; it is an ordered set of order policies which cover the demand up to t

Z : The objective value of the node; it is the sum of inventory holding cost, ordering cost and purchasing cost, including resale income, of the orders in P^t.

For every search tree, we consider $[1, 0, 0, f]$ as the root node. Each node in the search tree is formed by branching out another node. The branched node is called a "parent node" and the other one is called a "child node".

The algorithm presented for the problem in this paper has two main steps; branching step and fathoming step which are discussed in the following sections.

4.1. Branching step

Let $[k, I_{k-1}, Z, P^k]$, $k = 1, 2, ..., N$, be a node in the search tree such that $S'_k \neq f$ where $S'_k = \{(x, r) \in S_k \mid x - r + I_{k-1} \ge d_k\}$. For each $(x, r) \in S'_k$ a child node denoted by $[t, I_{t-1}, Z', P^j]$ is added to the set of children of $[k, I_{k-1}, Z, P^k]$ by the two following methods:

1- For $r = 0$ we have

$t = \max_{k+1 \le g \le N+1} \{ g \mid x + I_{k-1} \ge d_k^{g-1} \}$

$I_{t-1} = x + I_{k-1} - d_k^{t-1}$

$P^t = [P^k \, \mathrm{M} (x, 0)]$

$$Z' = Z + xp(x) + A + \sum_{i=k}^{t-1} h_i(x + I_{k-1} - d_k^i)$$

2- For $r > 0$ there exists v, $v = k, k+1, \ldots, N$, by Theorem 1 such that $r = r(k-1, v)$. In this case we have

$t = v + 1$

$I_{t-1} = 0$

$P^t = [P^k \cup \Phi(k-1, v)]$

$Z' = Z + Z_r(k-1, v)$

where $\Phi(k-1, v)$ is the set of order policies occurred through $\{k, k+1, \ldots, v\}$ and is obtained by (5).

4.2. Fathoming step

Using the two fathoming rules derived in this section, each child node generated in the previous step is checked for fathoming criteria in the fathoming step. If it is fathomed, then it is closed otherwise, it is considered as an open node for branching out in the next iteration of the algorithm. Furthermore, if the partial policy of the node is such that all periods demand are covered, it is fathomed and the current upper bound of the objective value is updated.

A common rule for fathoming a node in the branch and bound algorithms is to compare its objective value with the best current objective value. To make this fathoming more efficient in the minimization problems, a lower bound of the objective value of the policy which includes the node is compared with the best current objective value. The objective value in this problem has three elements; inventory holding cost, ordering cost and purchasing cost including resale income. The two following lemmas obtain a lower bound for these costs separately. Let $\bar{Z}$ be the sum of inventory holding cost and ordering cost of the orders occurred in the last N' periods starting from period t, $t = 1, 2, \ldots, N$, (obviously $N' = N - t + 1$). Furthermore, let $h = \min_{t \le i \le N}\{h_i\}$ and $d = \min_{t \le i \le N}\{d_i\}$. The following lemma which has been proved by Mirmohammadi et al. [5], obtains a lower bound for $\bar{Z}$.

Lemma 4. For $I_{t-1} = 0$, $t = 1, 2, \ldots, N$, a lower bound for $\bar{Z}$ is

$$\bar{Z}_L(N', d, h) = \begin{cases} N'A & A \le hd \\ \left\lfloor \dfrac{N'}{n^*} \right\rfloor \left(A + \dfrac{n^*(n^*-1)}{2} hd\right) + Ad(n') + \dfrac{n'(n'-1)}{2} hd & A > hd \end{cases} \tag{13}$$

where $n^* = \left\lfloor \dfrac{A}{hd} \right\rfloor + 1$, $n' = N' - \left\lfloor \dfrac{N'}{n^*} \right\rfloor n^*$ and $d(n') = \begin{cases} 1 & \text{if } n' > 0 \\ 0 & \text{if } n' = 0 \end{cases}$.

Lemma 5. The minimum purchasing cost including resale income of acquiring x units, $x \ge 0$, is $MPC(x)$ where

$$MPC(x)=\begin{cases} xU & \text{if} \quad x \le D_{0.5} \\ DU(1-a)-(D-x)C & \text{if} \quad D_{0.5} < x \le D \\ xU(1-a) & \text{if} \quad x > D \end{cases} \tag{14}$$

Proof. It follows directly by the definition of $D_{0.5}$. □

Now, by the two previous lemmas we can calculate a lower bound for the objective value of all policies derived from a special node in the search tree. If this lower bound exceeds our best current upper bound, the node will be fathomed. Let $[t, I_{t-1}, Z, P^t]$ be an open node in the search tree such that $1 \le t < N$. Furthermore, let $h' = \min_{t \le j \le N} \{h_j\}$ (the minimum unit holding cost per each period through t to the end of planning horizon), $d' = \min\{d_t - I_{t-1}, \min_{t+1 \le j \le N} \{d_j\}\}$ and Z_u be the best current upper bound for the objective value.

Fathoming rule 1. The node is fathomed if $Z + Z_L(N-t+1, d', h') + MPC(d_t^N - I_{t-1}) > Z_u$. Furthermore, when $t = N$ it is fathomed if $Z + A + MPC(d_N - I_{t-1}) > Z_u$

Proof. To reach a feasible order policy by this node, we need at least $d_t^N - I_{t-1}$ units of material. The lower bound of the purchasing cost including resale income of this amount of materials is $MPC(d_t^N - I_{t-1})$. Also by Lemma 4, the lower bound of the sum of inventory holding cost and ordering cost is $Z_L(N-t+1, d', h')$ from period t to period N. Note that the minimum demand and the minimum holding cost per unit in this interval is d' and h', respectively. Therefore, the objective value of all child nodes of $[t, I_{t-1}, Z, P^t]$ will increase at least by $Z_L(N-t+1, d', h') + MPC(d_t^N - I_{t-1})$. If $Z + Z_L(N-t+1, d', h') + MPC(d_t^N - I_{t-1}) > Z_u$, the optimal order policy of the problem can not be found by branching this node. Fathoming rule 1 is obviously true when $t = N$. □

Another way to fathom a node is to compare it with the existing nodes in the search tree. When two nodes end to a period with the same inventory, branching out the node with larger objective value will not result in the optimal order policy of the problem and it is fathomed. In the presented algorithm most of the nodes end to a period with the same inventory and simply one of them is fathomed, but in some cases their inventories are not equal. However, we can calculate an upper bound for the cost incurred while making their inventories equal. Now, increase the inventory of $[t, I_{t-1}, Z, P^t]$ by I, $2 \le t < N$, $0 \le I \le d_t - I_{t-1}$, by changing one of the orders in P^t (without any new ordering). Lemma 6 presents an upper bound for the cost incurred in this situation.

Lemma 6. An upper bound for the cost incurred by adding I more units to the inventory of $[t, I_{t-1}, Z, P^t]$ without any new ordering is $MI(P^t, I)$ where

$$MI(P^t, I) = \min_{(x_i, r_i) \in P^t} \left\{ MPC(x_i - r_i + I) - x_i p_i(x_i) + Cr_i + \sum_{j=i}^{t-1} Ih_j \right\} \tag{15}$$

where i is the period in which the ordering of (x_i, r_i) has occurred.

Proof. Omit each order (x_i, r_i) in P^t and acquire $x_i - r_i + I$ at the beginning of period i. Calculate the change of the total cost. □

Lemma 7 implies a property of $\frac{MPC(x)}{x}$ which let us to present Fathoming Rule 2.

Lemma 7. $\frac{MPC(x)}{x}$ is non-increasing in *x*.

Proof. It is obvious for $x \geq D$ and $x \leq D_{0.5}$. By calculating $\frac{MPC(x+1)}{x+1} - \frac{MPC(x)}{x}$ for $D_{0.5} < x < D$ the result is obtained.□

Fathoming Rule 2. Consider $[t, I_{t-1}, Z, P^t]$ as node 1 and $[t, I'_{t-1}, Z', P'^t]$ as node 2, $t = 2, 3, \ldots, N$, in a search tree. There are two possible cases for I_{t-1} and I'_{t-1} as stated below:

2.1. $I_{t-1} = I'_{t-1}$: In this case the node with larger objective value is fathomed. If their objective values are equal, one of them is fathomed arbitrarily.

2.2. $I_{t-1} \neq I'_{t-1}$: In this case one of them is larger than the other one. Let $I_{t-1} > I'_{t-1} \geq 0$ and $I = I_{t-1} - I'_{t-1}$

2.2.1. Node 2 is fathomed if $Z \leq Z'$.

2.2.2. Node 1 is fathomed if $Z' + I\frac{MPC(d_t - I'_{t-1})}{d_t - I'_{t-1}} < Z$.

2.2.3. Node 1 is fathomed if $Z' + MI(P'^t, I) < Z$, where $MI(P'^t, I)$ is obtained by Eq.(15).

Proof. Since 2.1 and 2.2.1 are obvious, we just consider 2.2.2 and 2.2.3. To prove 2.2.2 let *CH1* be any arbitrary child node branched out from node 1 by an order like (x_t, r_t). *CH1* ends to a period like k, $k = t+1, t+2, \ldots, N$, with I_{k-1} inventory such that $I_{k-1} = x_t - r_t + I_{t-1} - d_t^{k-1}$. The objective value of *CH1* is

$$Z + A + MPC(d_t^{k-1} + I_{k-1} - I_{t-1}) + \sum_{i=t}^{k-1}(d_t^{k-1} + I_{k-1} - d_t^i)h_i \tag{16}$$

Construct *CH2* by branching out node 2 on $(x_t + I, r_t)$. *CH2* ends to period k with exactly I_{k-1} inventory. The objective value of *CH*2 is

$$Z' + A + MPC(d_t^{k-1} + I_{k-1} - I'_{t-1}) + \sum_{i=t}^{k-1}(d_t^{k-1} + I_{k-1} - d_t^i)h_i \tag{17}$$

Let $Y = d_t^{k-1} + I_{k-1} - I_{t-1}$. We show that $MPC(Y + I) - MPC(Y) < Z - Z'$ by Eqs. (16) and (17). Obviously, $MPC(Y + I) - MPC(Y)$ is equal to

$Y(\frac{MPC(Y+I)}{Y+I} - \frac{MPC(Y)}{Y}) + I\frac{MPC(Y+I)}{Y+I}$ which is less than or equal to $I\frac{MPC(Y+I)}{Y+I}$ by Lemma 7. Since $Y+I \geq d_t - I'_{t-1}$, we have $I\frac{MPC(Y+I)}{Y+I} \leq I\frac{MPC(d_t - I'_{t-1})}{d_t - I'_{t-1}}$ which is less than $Z - Z'$ by the assumption of 2.2.2 and the proof is complete for this case. To prove 2.2.3, note that P^t has covered the demands of periods 1 to $t-1$ with I_{t-1} surplus inventory at the cost of Z. On the other hand, $Z' + MI(P'^t, I) < Z$ states that by changing one of the previous order policies in P'^t, it is possible to cover the demands of periods 1 to $t-1$ with I_{t-1} surplus inventory at a lower cost than Z. Therefore, the optimal order policy of the problem will not be found by continuing of P^t and the proof is complete. □

To illustrate the presented algorithm, an instance of the problem with five periods is considered with the relevant periods demand given in Table 1. The discount level is $D = 150$ and discount rate is $a = 20\%$.

Table 1. The periods demand of the example

t	1	2	3	4	5
d_t	50	80	60	100	40

The ordering cost and holding cost per unit per each period is 100 and 1, respectively. The net unit purchasing price and the unit resale price are 10 and 6, respectively; therefore, $D_{0.5}$ becomes 75 for this example. Figure 2 shows the search tree and its nodes for this example. Each node in Figure 2 consists the data for $[t, I_{t-1}, Z, P^t]$ which are defined in Section 4.1.

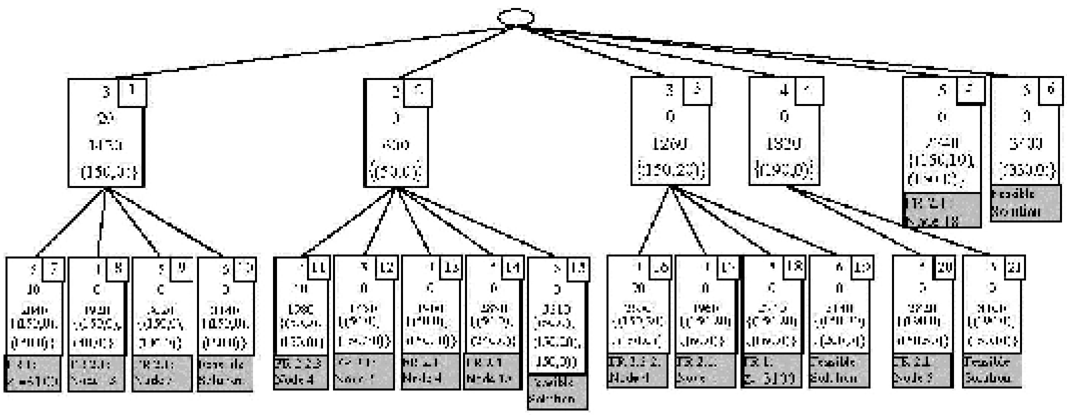

Figure 2. The search tree of the example

The shaded part of each node contains the index of the fathoming rule and the index of the node which is used in the fathoming rule. For example the expression "FR2-1, Node 4" in the shaded part of node 13 shows that node 13 has been fathomed by comparing with node 4 using fathoming rule 2-1. The number that is in the north east corner of each node is Node Formation Number (NFN) which shows the sequence of node generation in the branching step. The branching of nodes is based on the Width-First-Search. Note that before a node is branched, it is checked for fathoming.

5. THE TIME COMPLEXITY FUNCTION OF THE ALGORITHM

The number of nodes branched out to find the optimal order policy determines the running time of the algorithm. In other words, the worst-case time complexity function of the algorithm can be calculated by enumerating the maximum number of branched nodes in the search tree. By defining a graph, called "counter graph", corresponding to the search tree of a problem, we are able to enumerate the maximum number of branched nodes in an instance with N periods.

5.1. The definition of the counter graph

Let $T=(V,E)$ be the counter graph corresponding to the search tree of an instance with N periods. The set of vertices, V, and the set of edges, E, are defined as follow:

1. The vertex v labeled by (t, I_{t-1}), $t=1,2,...,N$, belongs to V if and only if there is a node in the search tree labeled by $[t, I_{t-1}, Z, P^t]$ which is branched out (i.e. it has at least one child). The corresponding node of v is $[t, I_{t-1}, Z, P^t]$ and the corresponding vertex of the node $[t, I_{t-1}, Z, P^t]$ is v. The corresponding vertex of the root node $[1,0,0,f]$ is denoted by v_0 and is called the root of $T=(V,E)$.
2. Let $v' \in V$ labeled by $(t', I_{t'-1})$, $t'=1,2,...,N,$, be the corresponding vertex of node $[t', I_{t'-1}, Z', P^{t'}]$ in the search tree.

2-1- $\overrightarrow{v_0 v'}$ labeled by 0, belongs to E if and only if $I_{t'-1}=0$.

2-2- $\overrightarrow{vv'}$ labeled by $x_t - r_t$, belongs to E if and only if $I_{t'-1}>0$ where v is the corresponding vertex of the parent node of $[t', I_{t'-1}, Z', P^{t'}]$ and (x_t, r_t) is the order policy by which $[t', I_{t'-1}, Z', P^{t'}]$ is derived from its parent.

When node $[t', I_{t'-1}, Z', P^{t'}]$ which has a parent node like $[t, I_{t-1}, Z, P^t]$ in the search tree ($0 \le t < t' \le N$) is branched out, its corresponding vertex v' is added to V. If this node has no beginning inventory, i.e. $I_{t'-1}=0$, v' is connected to the root of the counter graph, v_0, by an edge labeled by 0. If $I_{t'-1}>0$ then v' is connected to the v by an edge labeled by $x_t - r_t$ where v is the corresponding vertex of $[t, I_{t-1}, Z, P^t]$ and (x_t, r_t) is the order policy by which $[t', I_{t'-1}, Z', P^{t'}]$ is derived from $[t, I_{t-1}, Z, P^t]$, i.e. $P^{t'}=[P^t \,\mathbb{M}(x_t, r_t)]$. Since $x_t - r_t > 0$, each edge in $T=(V,E)$ labeled by 0 is a pendant edge of v_0. Figure 3 depicts the corresponding counter graph of the search tree shown in Figure 2.

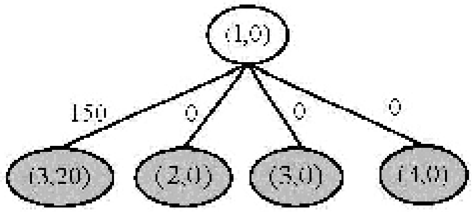

Figure 3. The counter graph of the example

5.2. The properties of the counter graph

The following lemmas characterizes $T=(V,E)$ more precisely which let us to present an upper bound for $|V|$.

Lemma 8. Let in, v,v' labeled by x, $x\geq 0$, be an arbitrary edge of $T=(V,E)$. Then $x\in\{0,D\}$.

Proof. Let $[t,I_{t-1},Z,P^{t}]$ and $[t',I_{t'-1},Z',P^{t'}]$ be the corresponding nodes of v and v', respectively. If $I_{t'-1}=0$ then $v=v_0$ and by the definition of $T=(V,E)$, $x=0$ and the proof is complete. Let $I_{t'-1}>0$. The ordering occurred at t is in $P^{t'}$, i.e. $(x_t,r_t)\in P^{t'}$. If $r_t>0$, then $I_{t'-1}=0$ by part 2 of Section 4-1 which contradicts $I_{t'-1}>0$. For $r_t=0$, we have $x_t=d_t^{t'}-I_{t-1}$ or $x_t=D$ by Theorem 1. The first case contradicts $I_{t'-1}>0$ and the second one imposes $x=D$ therefore, the proof is complete for $I_{t'-1}>0$. □

Lemma 9. Each vertex is unique in $T=(V,E)$.

Proof. Assume on the contrary that a counter graph $T=(V,E)$ has two vertices like $v,v'\in V$ which are labeled by (t,I_{t-1}), $1\leq t\leq N$, $I_{t-1}\geq 0$. Since the corresponding nodes of v and v' end to t with equal inventory I_{t-1}, one of them is fathomed by Fathoming Rule 2.1 and it is never branched. Hence, either $v\notin V$ or $v'\notin V$ and the proof is complete. □

Lemma 10. Each vertex, except the root of $T=(V,E)$, has at most one child.

Proof. Assume on the contrary that there is a vertex $v\in V$, $v\neq v_0$, which has two or more pendant edges with different labels. Let a and b be their labels. Since $a\neq b$ and $a,b\in\{0,D\}$ by Lemma 8, one of them is 0 and the other one is D.But it is impossible because the edge labeled by 0 is a pendent edge of the root of v_0 by the definition of $T=(V,E)$ and the proof is complete. □

Lemma 11. The root of $T=(V,E)$, v_0, has at most N children.

Proof. The number of pendant edges of v_0 determines the number of its children. The number of pendant edges of v_0 labeled by a positive number x, is at most 1 by Lemma 8. By the definition of $T=(V,E)$, the number of nodes with zero inventory branched out in the search tree, determines the number of pendant edges of v_0 labeled by 0. From all nodes in the search tree with zero inventory which are ended to period k, $2\leq k\leq N$, at most one node is branched out and the other nodes are fathomed by Fathoming Rule 2-1. Since $2 \le k \le N$, there are at most $N-1$ nodes with zero inventory which are branched in the search tree; hence, the maximum number of pendant edges of v_0 is $N-1+1$ in $T=(V,E)$ and the proof is complete. □

Lemma 12. Let H be the height of $T=(V,E)$, then $H \le \min\left\{\left\lceil \frac{d_1^N}{D} \right\rceil, N\right\}$ and $|V| \le NH+1$.

Proof. Let v labeled by (t, I_{t-1}), $t \le N, I_{t-1} \ge 0$, be a leaf of T which has the longest path from the root of T. Let this path and its length be denoted by LP and $|LP|$, respectively, then $H=|LP|$. Since each edge in LP represents an ordering and each ordering covers the demand of at least one period therefore, $H=|LP| \le N$ .On the other hand since $t \le N$, we have

$$d_1^{t-1} + I_{t-1} < d_1^N \qquad (18)$$

Since by Lemma 8 each edge of LP except the root pendant edge is labeled by D , we have $(|LP|-1)D \le d_1^{t-1} + I_{t-1}$. Therefore, by (19) and the fact that $|LP|$ is an integer we have $|LP| \le \left\lceil \frac{d_1^N}{D} \right\rceil$ and $H \le \min\{\left\lceil \frac{d_1^N}{D} \right\rceil, N\}$. Now, let $R(i)$, $i=0,1,2,...,H$, be the total number of the vertices in the i^{th} level of T. We have $R(0)=1$ and $R(i) \le N$, $i=1,2,...,H$, by Lemma 11 and Lemma 10. Therefore, $|V| = R(0) + \sum_{i=1}^{H} R(i) \le NH+1$ and the proof is complete. □

Without loss of generality, the worst-case time complexity function of the algorithm with only Fathoming Rule 2-1 and Fathoming Rule 1 is considered here, since these rules are more effective than the other ones. However, the algorithm with all fathoming rules is not more complex than what is considered here.

For the rest of this section we need to define the following notations.

NBN : The maximum number of branched nodes

MCN : The maximum number of the children of a branched node

t_B : The maximum time required for generating a child node

t_{F1} : The maximum time required for checking a node by Fathoming Rule 1

t_{F2} : The maximum time required for checking a node by Fathoming Rule 2.1.

We know that by Lemma 12, $NBN \le N^2+1$ and by Eq. (10) $S_k \le N+1$ for $k=1,2,...,N$, therefore, $MCN \le N+1$. Furthermore, all parameters t_B, t_{F1}, t_{F2} are constant and independent of the problem parameters.

5.3. The worst case time complexity function

By the properties of the counter graph described in the above lemmas Theorem 2 is followed which presents an upper bound for the worst-case time complexity function of the presented algorithm.

Theorem 2. Let $f(N)$ be the worst-case time complexity function of an instance of the problem with N periods. Then, $f(N) = O(N^3)$.

Proof. The maximum total number of nodes generated in the search tree to find the optimal order policy in an instance with N periods is $NBN \times MCN \le (N^2+1)(N+1)$. On the other hand, the maximum time spent for each node in the search tree is $t_B + t_{F1} + t_{F2}$ which is constant. Therefore, $f(N) \le (N^2+1)(N+1)(t_B + t_{F1} + t_{F2})$ where the last term is $O(N^3)$ and the proof is complete. □

6. COMPUTATIONAL EXPERIENCE

In order to demonstrate the computational efficiency of our algorithm, it has been coded in C++ 6.0 and the average CPU times required for solving some randomly generated problems have been gathered.

6.1. Experimental design

For the experimental design, we have adopted the framework of control factors in Mirmohammadi et al. [5]. There are two factors which characterize the demand environment of the problems, the number of periods in the planning horizon (*N)* and the coefficient of variation of demand *(CV)*. The coefficient of variation *(CV)* measures the period-to-period variation in demand. It is the ratio of the standard deviation of the demand to the average demand. The *CV* values used for this experiment are 0.29 and 1.85. The values of *N* are 24, 124, 224,…, 924 and 1024. The ratio of the discount level to the average demand (*D/R*) is set to 2 and the discount rate, a , is set to 10%. The ratio of the unit resale price to discounted unit net price ($AR = \frac{C}{(1-a)U}$) describes the attractiveness of resale and has the values of 0.15, 0.3, 0.45, 0.6 and 0.75. The values of other parameters used in this experiment are listed in Table 2.

Table 2. Values of parameters used in the experiment

Ordering cost (A) =92
Inventory holding Cost $h_j = 2 / period / unit, \forall j = 1, 2, ..., 5$
unit net price $U = 500$
Average demand (R) =92

For each combination of *N*, *CV* and AR, 20 instances are randomly generated from a truncated normal distribution with mean of 92 and variance obtained by *CV* parameter, to provide 2200 test problems for the experiment. The performance criterion is the CPU time (m. sec.) on the Pentium(R) 4 CPU 3.41 GHz with 2.00 GB of RAM. Table 3 contains the results.

Table 3. Average CPU time of each combination of *N*, *CV* and *AR*

N	***AR*=0.15**		***AR*=0.3**		***AR*=0.45**		***AR*=0.6**		***AR*=0.75**	
	***CV*=0.29**	***CV*=1.85**	***CV*=0.29**	***CV*=1.85**	***CV*=0.29**	***CV*=1.85**	***CV*=0.29**	***CV*=1.85**	***CV*=0.29**	***CV*=1.85**
24	0.75	0	0	0.75	0	0.75	0	0.75	0	1.55
124	71.05	56.25	73.4	59.35	69.55	55.45	71.05	56.2	71.85	56.25
224	408.65	347.65	425	348.45	435.95	347.65	423.45	352.3	419.4	348.4
324	1243.65	1017.1	1256.2	1022.75	1256.6	1039.05	1252.25	1252.25	1262.45	1029.65
424	2953.2	2380.5	2895.45	2397.65	2907.8	2375.05	2899.1	2384.35	2889.8	2378.05
524	5575.9	4569.45	5536.15	4526.6	5524.75	4561.6	5532.05	4516.45	5566.3	4549.95
624	9479.65	7738.25	9640	7709.3	9512.5	7774.3	9481.8	7722.6	9446.1	7734.5
724	15075.7	12280.45	15488.95	12071.15	14819.75	12306.35	15017.1	12050.7	14992.8	12194.6
824	22506.15	18051.5	22611.5	18376.4	22457.05	18029.55	22446.65	18048.65	22391.55	18060
924	31912.45	25758.65	32254.8	26017.9	32235.1	25184.25	32065.56	25613.4	31869.45	25645.3
1024	44316.45	35848.35	44264.2	35614.85	44381.15	35533.6	44268.7	35807.65	44428.2	35138.95

6.2. Analysis of test results

The following summarized conclusions are derived from the experiment:

1. When N increases constantly from 0 to 1024, the CPU time increases with the maximum order of 3. In other words, the time complexity function of the algorithm when N changes from 0 to 1024 is $O(N^3)$ in this experiment which confirms Theorem 2. The CPU time and the equation of the trend line is depicted in Figure 4 for *CV*=0.29 and $AR = 0.15$.
2. The CPU time decreases when *CV* increases from 0.29 to 1.85 for each level of *N* and AR.
3. It seems that the attractiveness of resale has no significant effect on CPU time since the CPU time has no stable pattern versus the variation of AR.

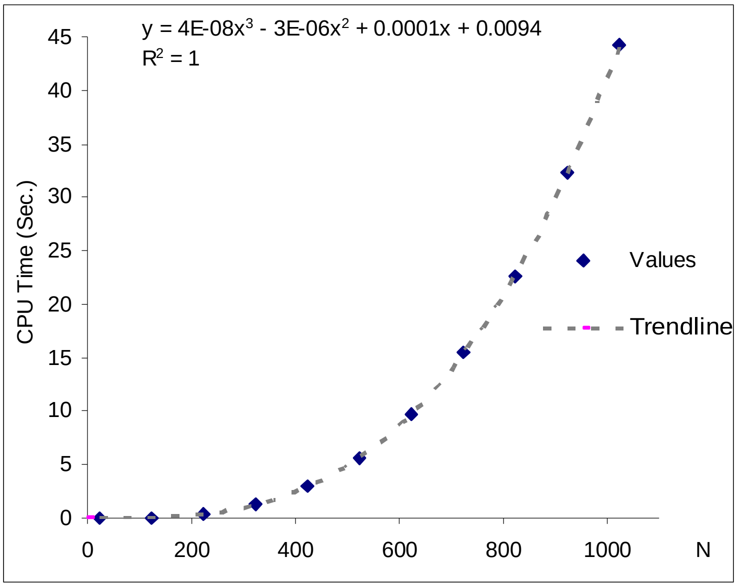

Figure 4. CPU time increase with CV=0.29 and $AR = 0.15$

7. CONCLUSIONS

The properties of an optimal order policy for the dynamic quantity discount problem with resale in single price break points have been discussed in this paper. On the basis of them, an optimal algorithm based on branch and bound approach has been presented for the problem. By defining a rooted tree graph corresponding to each search tree called counter graph, it has been shown that the worst-case time complexity function of the presented algorithm is $O(N^3)$. The efficiency of the presented algorithm is shown by solving 2200 randomly generated problems. Experimental results confirm that the time complexity function of the presented algorithm is $O(N^3)$ for the adjusted parameters in the experiment.

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