PEIECWISE CONSTANT LEVEL SET METHOD BASED FINITE ELEMENT ANALYSIS FOR STRUCTURAL TOPOLOGY OPTIMIZATION USING PHASE FIELD METHOD

S. Shojaee[*][†], A. Mohaghegh and A. Haeri
Department of Civil Engineering, Shahid Bahonar University of Kerman, Kerman, Iran

ABSTRACT

In this paper the piecewise level set method is combined with phase field method to solve the shape and topology optimization problem. First, the optimization problem is formed based on piecewise constant level set method then is updated using the energy term of phase field equations. The resulting diffusion equation which updates the level set function and optimization problem is solved through finite element method. The proposed method enhances the convergence rate and solution efficiency. Various two-dimensional examples are solved to verify the performance of proposed method.

1. INTRODUCTION

The shape and topology optimization of structures is one of the most important issues in various engineering applications. Significant development in shape and topology optimization methods is achieved by researchers in recent decades [1-5].

Level set method introduced by Osher and Sethian [6-8], has been successfully used in shape and topology optimization of structures. In level set method the inner and outer boundaries are considered as design variables and the structural boundaries are defined by zero level of level set function. By this approach the domain boundaries can be easily combined or separated from each other. If one uses the explicit methods for solution of Hamilton-Jacobi equation, some restrictions will arise, like time consuming process of

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[*]Corresponding author: S. Shojaee, Department of Civil Engineering, Shahid Bahonar University, Kerman, Iran
[†]E-mail address: saeed.shojaee@uk.ac.ir (S. Shojaee)

initialization, satisfaction of Courant-Friedrichs-Lewy (CFL) condition and dependence of final topology to initial guess.

Piecewise Constant Level Set (PCLS) method was first introduced by Lie et al. [9], for interface problems such as image processing. In piecewise constant level set method unlike discrete level set method, there is no need to solve the Hamilton-Jacobi equation, thus it is free of the CFL condition and the reinitialization scheme. This feature will result in significant reduction of time cost. In the PCLS method the interface is determined by forcing the value of the LSF at each mesh point to be one of the piecewise constant values. Therefore it enables this method to create holes during the evolution of the LSF without using topology derivatives.

Recently the PCLS method is used with Lagrange multiplier method in constraint minimization problem [10], however in Lagrange multiplier method the iterations number for convergence is relatively high.

Shojaee and Mohammadian [11] combined PCLS with Merriman-Bence-Osher (MBO) scheme for topology optimization problems which resulted in increase of convergence rate. In present study the phase field method is applied for increase of convergence rate and reducing the consumed time of optimization process.

The phase field method and its concepts were developed by Cahn and Hilliard [12] and Allen and Cahn [13]. Bourdin and Chambolle [14] proposed the idea of implementation of phase field method in structural optimization. Takezawa et al. [15] adopted the phase field method and sensitivity analysis in solution of topology optimization problems.

In this study the PCLS method is combined with phase field method to solve the topology optimization problems. In details, PCLS is used to form the optimization problem which is updated using the energy term of phase field equations. The resulting diffusion equation which updates the level set function and optimization problem is solved through finite element method. The comparison of proposed method with references confirms its convergence superiority and solution efficiency.

This paper is organized as follows: in Sec. 2 PCLS basics are briefly reviewed. Sec. 3 deals with PCLS in optimization platform. The concepts of pahse field are described in Sec. 4. In Sec. 5 PCLS and phase field are combined and consequently the solution of resulting differential equiation using finite element method is described in Sec 6. The proposed method is verified with refrence methods in Sec. 7 through solving some 2D optimization problems.

2. A PIECEWISE CONSTANT LEVEL SET METHOD

In this section the piecewise constant level set (PCLS) method is briefly reviewed [11].

Consider partitioning the domain Ω into n sub domains $\{\Omega\}_{i=1}^{n}$ as follows:

$$\Omega = \bigcup_{i=1}^{n} \Omega_i \cup \Gamma \qquad (1)$$

where Γ is the union of the boundaries of the sub regions.,

A piece-wise constant function, $\phi : \Omega \to R$, can be defined on the open and bounded domain which takes the following values:

$$\phi(x) = i, \qquad x \in \Omega_i, i = 1, 2, ..., n \tag{2}$$

Thus, for any given partition $\{\Omega\}_{i=1}^n$ of the domain Ω we just need one PCLS function ϕ which takes the values of $1, 2, ..., n$.

A characteristic function, $\psi_i(x)$, for each subdomain, Ω_i, can be defined as follows:

$$\psi_i = \frac{1}{\alpha_i} \prod_{j=1, j \neq i}^n (\phi - j) \tag{3}$$

$$\alpha_i = \prod_{k=1, k \neq i}^n (i - k) \tag{4}$$

Therefore, $\psi_i(x) = 1$ for $x \in \Omega_i$, and $\psi_i(x) = 0$ wherever Eq. (2) holds. To represent the different properties in each sub domain, $\rho = c_i$ in Ω_i, we define a piecewise density function as follow,

$$\rho(\phi) = \sum_{i=1}^n c_i \psi_i(\phi) \tag{5}$$

Where

$$K(\phi) = (\phi - 1)(\phi - 2)...(\phi - n) = \prod_{i=1}^n (\phi - i) \tag{6}$$

A piecwise constant constraint is defined to avoid vacuum or overlap between the different phases:

$$K(\phi) = 0 \tag{7}$$

finally one can calculate the volume and the perimeter of individual subdomains in the following form:

$$|\Omega_i| = \int_\Omega \psi_i dx \tag{8}$$

$$|\partial \Omega_i| = \int_\Omega |\nabla \psi_i| dx, \tag{9}$$

3. PIECEWISE CONSTANT LEVEL SET FRAMEWORK FOR TOPOLOGY OPTIMIZATION PROBLEM

Here a simple framework of PCLS for topology optimization is reviewed [16].

With respect to phase field, PCLS is implemented in two phases: 0 and 1. The constant piecewise density function is defined in two phases as follows:

$$\rho(\phi) = -c_1(\phi - 1) + c_2\,\phi^2 \tag{10}$$

Where c_1 and c_2 are prescribed values respectively for hollow and solid parts. Now if one considers $c_1 = 0$ for hollow phase and $c_2 = 1$ for solid phase, the density function will have the following relationship with PCLS function.

$$\rho(\phi) = \begin{cases} c_1 & \phi = 0 \\ c_2 & \phi = 1 \end{cases} \tag{11}$$

A piecewise constant constraint should be defined to guarantee the convergence of level set function, φ, to a unique value:

$$k(\varphi) = 0, \ \ k(\varphi) = \varphi^2(\varphi - 1) \tag{12}$$

This indicates that every point in the design domain must belong to one and only one phase and there is no overlap and vacuum between different phases.

In this paper the optimization objective is to minimize the compliance over the structural domain for general loading condition and various boundary conditions. In other words the optimization problem is defined as follows:

$$\begin{aligned} \min \ \ & j(\mathrm{u}, \varphi) = \int_\Omega \rho(\varphi) F(\mathrm{u})\,\mathrm{d}\Omega + \beta \int_\Omega |\nabla\varphi|\,\mathrm{d}\Omega \\ s.t \ \ & H_1 = \int_\Omega \rho(\varphi)\,\mathrm{dx} - V_0 \leq 0 \\ & H_2 = K(\varphi) = 0 \\ & a(\mathrm{u}, v, \varphi) = l(v, \varphi) \\ & a(\mathrm{u}, v, \varphi) = \int_\Omega \rho(\varphi) E_{ijkl}\varepsilon_{kl}(v)\,\mathrm{d}\Omega \\ & l(v, \varphi) = \int_\Omega f.v\,d\Omega + \int_{\Gamma_N} p.v\,d\Gamma \end{aligned} \tag{13}$$

where Ω is the structural domain and its boundary is represented by $\Gamma = \partial\Omega$. Also in the

linear elastic equilibrium equation, u denotes the displacement field, u_0 is the prescribed displacement on Γ_D, E_{ijkl} is the elasticity tensor, ε_{ij} is the strain tensor. f and p are body force and surface load respectively. In the objective function $J(u)$, the first term is the mean compliance where functional $F(u)$ is the strain energy density and ρ is the material density ratio. The second term in the objective function is the regularization term. β is a non-negative value to control the effect of second term. Indeed, this term controls both the length of interfaces and the jump of φ, because the value of φ may not be continuous in the PCLS. H_1 defines the material fraction for different phases and V_0 is the maximum allowable volume of the design domain. H_2 is the piecewise constant constraint to guarantee that the level set function belongs to only one phase. If we use the augmented Lagrangian method to convert Eq. (13) into an unconstrained one, the following form is obtained:

$$
\begin{aligned}
L(\varphi,\lambda) &= J(\varphi) - a(\mathrm{u},\upsilon,\varphi) + \mathrm{l}(\upsilon,\varphi) + \lambda_1 H_1 \\
&+ \frac{1}{2\mu_1} H_1^2 + \lambda_2 \int_\Omega H_2 \, d\Omega + \frac{1}{2\mu_2} \int_\Omega H_2^2 \, d\Omega
\end{aligned}
\tag{14}
$$

where $\lambda_1 \in R$ and $\lambda_2 \in L^2(\Omega)$ are Lagrange multipliers and $\mu_1, \mu_2 > 0$ are penalty parameters. Now, we need to find a saddle point of the augmented Lagrangian functional L. To find the saddle point of this function where there is no body force, f, we have the following equation as suggested by Wei and Wang [17]:

$$
\int_\Omega \psi(\mathrm{u},\varphi,\tilde{\lambda}_1,\tilde{\lambda}_2).\delta\varphi \, d\Omega = 0
\tag{15}
$$

$$
\begin{aligned}
\psi(\mathrm{u},\varphi,\tilde{\lambda}_1,\tilde{\lambda}_2) &= -\frac{1}{2}\rho'(\varphi)\, \mathrm{E}_{ijkl}\, \varepsilon_{ij}(\mathrm{u})\varepsilon_{kl}(\mathrm{u}) \\
&+ \beta\nabla.(\frac{\nabla\varphi}{|\nabla\varphi|}) + \tilde{\lambda}_2\, \mathrm{K}'(\varphi)
\end{aligned}
\tag{16}
$$

$$
\rho'(\varphi) = \frac{\partial\rho(\varphi)}{\partial\varphi} = \frac{1}{2}(c_2 - c_1)
\tag{17}
$$

$$
\tilde{\lambda}_2 = \lambda_2 + \frac{1}{\mu_2}K(\varphi)
\tag{18}
$$

$$
K'(\varphi) = \frac{\partial K(\varphi)}{\partial\varphi} = 2\varphi
\tag{19}
$$

$$
\tilde{\lambda}_1 = \tilde{\lambda}_1 + \frac{1}{\mu_1}\left(\int_\Omega \rho(\varphi)\, \mathrm{d}\Omega - V_0\right)
\tag{20}
$$

the steepest descent method can be applied to satisfy Eq. (13) [12]:

$$\frac{d\varphi}{dt} = -\psi \ , \varphi^0 = \varphi_0 \tag{21}$$

Thus, the optimization problem is transformed into an ordinary differential problem of initial value φ_0. The simplest approach for solving the Eq. (21) is to use an explicit scheme. However, here the phase field method is combined with PCLS method as described in section 5. In proposed method the piecewise constant term is substituted by energy term in phase field method. Therefore the H_2 constraint in Eq. (13) is omitted and there is no need to update μ_2 and λ_2 anymore.

4. PHASE FIELD METHOD

In this section the concepts of phase field method are discussed [15]. The phase field function $\phi(x)$ is defined on whole analysis domain to represent the phase of all local points within the domain. From the physical aspect; the phase field alters the mean phase of local points. Consider a system composed of two phases, α and β The boundary of each phase is represented like a norm function which is extrapolated among different values. This term is called the diffuse interface. The free energy of Van Der Waals system is given by [15]:

$$F(\phi) = \int_\Omega (\frac{\varepsilon}{2}|\nabla \phi|^2 + \varepsilon^{-1} f(\phi))dx \tag{22}$$

where $\varepsilon > 0$ is a coefficient which checks influence of each term. The first term of Eq. (22) depicts the interaction energy of domain in main theory, and the second term depicts the double well potential with value of $f'(\alpha) = f'(\beta) = 0$. Double well potential implies the existence of lower values of free energy with minimum response to each phase. The time-dependent evolutionary equations are introduced in the following. The variation of field phase function is assumed linearly dependent to the direction that the free energy function is minimized:

$$\frac{\partial \phi}{\partial t} = -M(\phi)\frac{\delta F(\phi)}{\delta \phi} \tag{23}$$

By replacing Eq. (22) in Eq. (23):

$$\frac{\partial \phi}{\partial t} = M(\phi)(\varepsilon \nabla^2 \phi - \varepsilon^{-1} f'(\phi)) \tag{24}$$

Eq. (24) is known as Allen-Cahn Equation [15].

4.1 Domain defined by phase field function

Here the optimization domain, D, is divided into two phases, Ω_0 and Ω_1, and the boundary between phases is denoted by ξ which is called diffuse interface region. The domain $\Omega_1 \, (x \in D \,|\, \phi(x) = 1)$ is related to the optimal shape Ω and $\Omega_0 \, (x \in D \,|\, \phi(x) = 0)$ is related to $D \backslash \Omega$. ξ is represented by an interpolation function between two phases. The domain of optimizer, D, includes all acceptable shapes of Ω.

$$\Omega \subset D \qquad (25)$$

Where:

$$\Omega \subset (\Omega_1 \cup \xi) \qquad (26)$$

$$D \backslash \Omega \subset (\Omega_0 \cup \xi) \qquad (27)$$

Therefore the phase field domain function ϕ is defined as follows:

$$\begin{cases} \phi = 1 \Leftrightarrow x \in \Omega_1 \\ 0 < \phi < 1 \Leftrightarrow x \in \xi \\ \phi = 0 \Leftrightarrow x \in \Omega_0 \end{cases} \qquad (28)$$
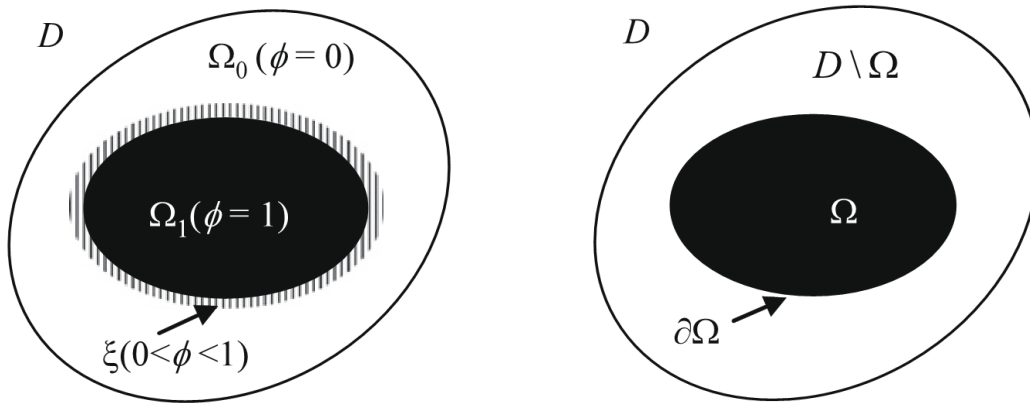
(a) The main domain   (b) The domain represented by phase field function
Figure 1. Phase field function domain [15]

Ω is the domain that the changes are applied on it during optimization process. A number of partial differential equations define this domain. The ∂D boundary of Ω domain is divided into two boundaries, ∂D_D with Diritchlet boundary condition and ∂D_N with Neumann boundary conditions. The solid phase of Ω_1 is depicted by martials of density

$\rho = c_2$ and the hollow phase is depicted by the density of $\rho = c_1$ where for $\rho(\phi)$:

$$\rho(\phi) = -c_1(\phi-1) + c_2\, \phi^2 \tag{29}$$

The main domain Ω is presented in the form of a union of Ω_1 and ξ. The location of $\partial\Omega$ boundary is unknown unless when it is located on the ξ. When ε is small enough, the diffuse interface region become extensively thin and this approximates the representation of $\partial\Omega$.

5. COMBINATION OF PIECEWISE CONSTANT LEVEL SET METHOD WITH PHASE FIELD METHOD

Now consider the solution of Eq. (13). In order to increase the convergence rate and generation of smooth and thin boundaries, the energy term of phase field equation is combined with Eq. (13). The obtained equation is a partial second order differential equation from Allen-Cahn equation [15]. Several methods are proposed to solve this differential equation. In this paper, the finite element method is applied to solve this equation, which results in accurate solutions and better generation of boundaries. Now we can define the minimization problem of strain energy using the energy term stated in phase field:

$$
\begin{aligned}
&\min \ \ j(\mathrm{u},\phi) = \int_\Omega \rho(\phi) F(\mathrm{u})\, \mathrm{d}\Omega + \beta \int_\Omega \left|\nabla\phi\right|^2 \mathrm{d}\Omega \\
&s.t \ \ H_1 = \int_\Omega \rho(\phi)\, \mathrm{dx} - V_0 \le 0 \\
&\mathrm{a}(\mathrm{u},v,\phi) = \mathrm{l}(v,\phi) \\
&\mathrm{a}(\mathrm{u},v,\phi) = \int_\Omega \rho(\phi) E_{ijkl}\, \varepsilon_{kl}(v)\, \mathrm{d}\Omega \\
&l(v,\phi) = \int_\Omega f.v\, d\Omega + \int_{\Gamma_N} p.v\, d\Gamma
\end{aligned} \tag{30}
$$

By adding this term to the above equation H_2 is omitted form spiecewise constant term which results in significant increase of convergence rate. Here the Lagrange method is used to convert the constrained optimization problem to an unconstrained one.

$$
\begin{aligned}
L(\phi,\lambda) &= J(\phi) - a(\mathrm{u},\upsilon,\phi) + \mathrm{l}(\upsilon,\phi) \\
&+ \lambda_1 H_1 + \frac{1}{2\mu_1} H_1^2
\end{aligned} \tag{31}
$$

$$\psi(\mathrm{u},\phi,\tilde{\lambda}_1,\tilde{\lambda}_2) = -\frac{1}{2}\rho'(\phi)\,\mathrm{E}_{ijkl}\,\varepsilon_{ij}(\mathrm{u})\varepsilon_{kl}(\mathrm{u})$$
$$+\beta\nabla^2\phi+\lambda_1\rho'(\phi)$$
(32)

$$\rho'(\phi)=\frac{\partial\rho(\phi)}{\partial\phi}=(-c_1+2c_2\,\phi)$$
(33)

$$\tilde{\lambda}_1=\tilde{\lambda}_1+\frac{1}{\mu_1}\left(\int_\Omega\rho(\phi)\,\mathrm{d}\Omega-V_0\right)$$
(34)

$$\frac{d\phi}{dt}=-\psi\;,\phi^0=\phi_0$$
(35)

$$\frac{d\phi}{dt}=-\beta\nabla^2\phi+\rho'(\phi)\left[\frac{1}{2}E_{ijkl}\varepsilon_{ij}\varepsilon_{kl}+\lambda_1\right]$$
(36)

To update the level set function ϕ the following explicit form is used:

$$\varphi^{n+1}=\varphi^n+\frac{d\varphi}{dt}.\Delta t$$
(37)

finally by substituting Eq. (35) in Eq. (36) the following equations are obtained:

$$\frac{\phi^{n+1}}{\Delta t}=\frac{\phi^n}{\Delta t}-\beta\nabla^2\phi+S$$
$$S=\rho'(\phi)\left[\frac{1}{2}E_{ijkl}\varepsilon_{ij}\varepsilon_{kl}+\lambda_1\right]$$
$$\phi=1\qquad on\;\;\partial\Omega$$
(38)

6. SOLUTION OF DIFFERENTIAL EQUATION USING FEM

Using FEM, the weak form of Eq. (38) can be written as follows:

$$\int_\Omega\frac{\phi^{n+1}}{\Delta t}\phi\,d\Omega=\int_\Omega\frac{\phi^n}{\Delta t}\phi\,d\Omega-\int_\Omega\nabla^T\phi^{n+1}(\beta\nabla\phi)d\Omega$$
$$+\int_\Omega S\phi\,d\Omega$$
$$S=\rho'(\phi)\left[\frac{1}{2}E_{ijkl}\varepsilon_{ij}\varepsilon_{kl}+\lambda_1\right]$$
$$for\forall\phi\in\phi$$
$$\phi=1\;on\;\partial\Omega$$
(39)

where:

$$\phi = \left\{ \phi(x) \mid \phi(\mathrm{x}) \in H^1(D) \, with \, \phi = 1 \, on \, \partial\Omega \right\} \tag{40}$$

Using FEM Eq. (39) can be rewritten as follows:

$$\left[\int_A \frac{1}{\Delta t} N^T N dA + \int_A \nabla N^T B \nabla N dA \right] \phi^{n+1} = \int_A \left(S + \frac{\phi^n}{\Delta t} \right) dA \tag{41}$$
$$\phi = 1 \; on \; \partial\Omega$$

where N is the shape function and β controls the stability of results. The appropriate selection of β may result in suitable shapes. Four-node square elements of size 1×1 are used.

7. NUMERICAL EXAMPLES

In this section some important issues about implementation of the proposed PCLS method with phase field method are discussed. These implementations are developed in order to improve the performance of the proposed method. The finite element analysis is based on 'ersatz material' scheme [11], which fills the void areas with one weak material. All numerical examples have the following data: Young's modulus of real material is assumed 1 and ersatz material 10^{-3}. This also means $c_2 = 1$ and $c_1 = .001$. Poisson' ratio for two materials is assumed 0.3 and the thickness is $t = 1$.
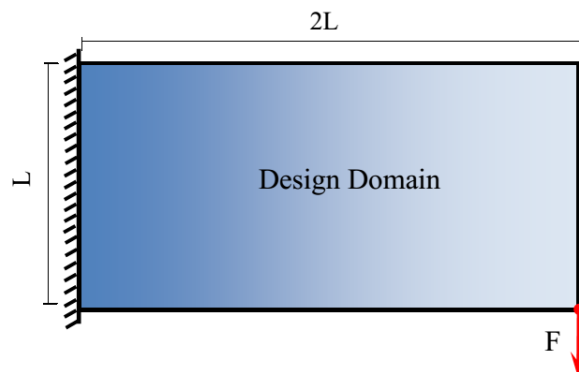
Figure 2: A cantilever beam

7.1 The cantilever beam 1

Fig. 2 shows the design domain of a cantilever beam. The boundary of the left side is fixed, and a vertical concentrated force $F = 1N$ is loaded at the bottom of its right free side. The size of the design domain is 80×40 with a squared mesh of size 1×1 and the volume

fraction is 50%. This example is solved by PCLS-PFM scheme. For this method the size of time step is $\tau = 46$ and the other parameters are considered as $\beta = 1e - 5, \lambda_1 = 0.01$, $\mu_1 = 700$. The difficult part is to find these parameters that can be chosen after testing different values for these parameters. Therefore, different values may lead to different optimal topologies. The evolution process of the optimal topology and the binary level set surface are shown in Fig. 3. Fig. 4 shows the convergence rate of the objective function and the volume ratio for the short cantilever beam. It can be seen that the compliance of the optimal solution is considerably better than that of the initial design and the compliance converges in a fast and stable way because of the present phase field method (PFM). The piecewise constant constraint is satisfied in each iteration and the PCLS takes the value 0 or 1 everywhere in the design domain. Therefore, there is no overlap and vacuum between the subdomains of different phases. In the conventional PCLS this constraint can be applied with the penalization method or augmented Lagrangian method. The penalty method is more stable and can be used easily, but to satisfy the constraint exactly, one has to set it very small. However, this may cause the instability of the numerical process. By using the augmented Lagrangian method, we do not need to use a small value for penalty parameter but the iteration process is high at convergence. Thus, by coupling the PCLS and the PFM scheme, we can release this constraint. It should be noted there is no need to use the penalty method or the augmented Lagrangian method for this constraint.

(a) Initial design

(b) Step 10

(c) Step 15

(d) Step 23

(e) Final design
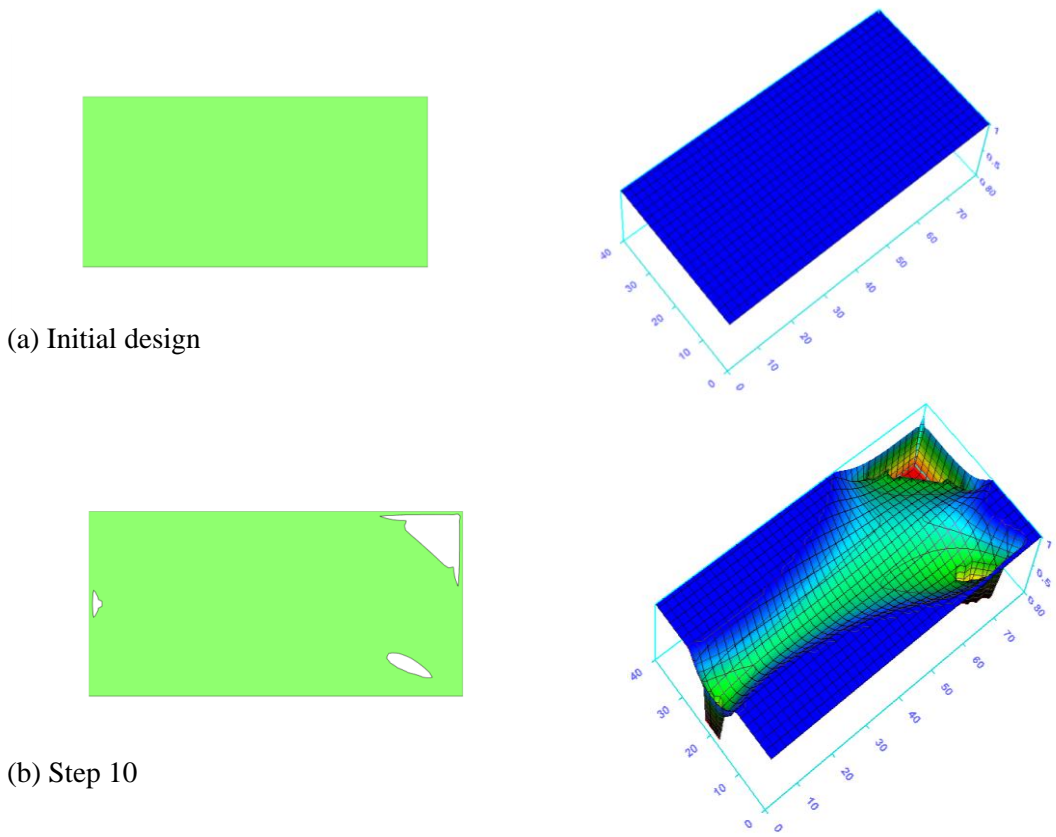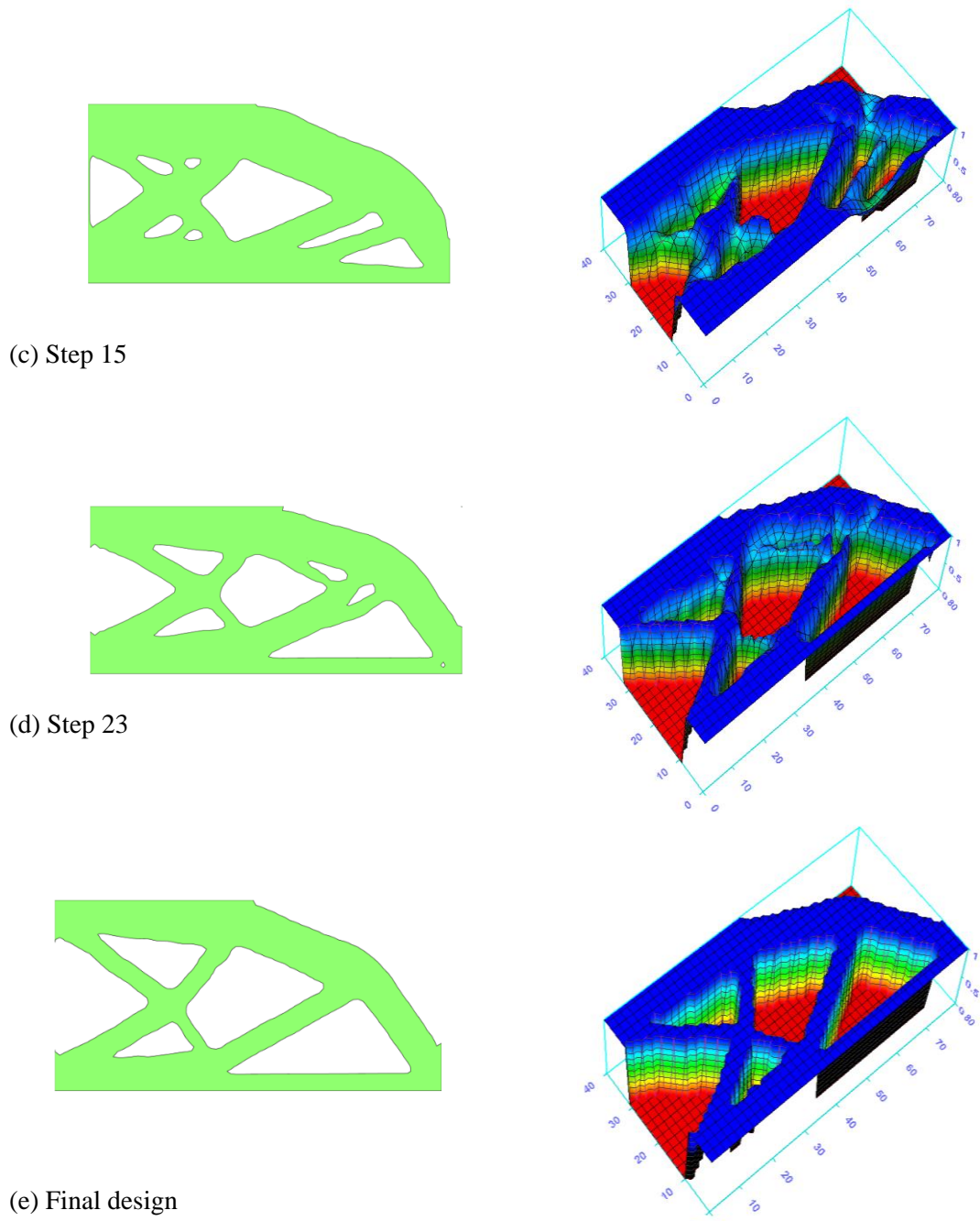
Figure 3. The evolution process of optimal design and the level set function with Phase field method
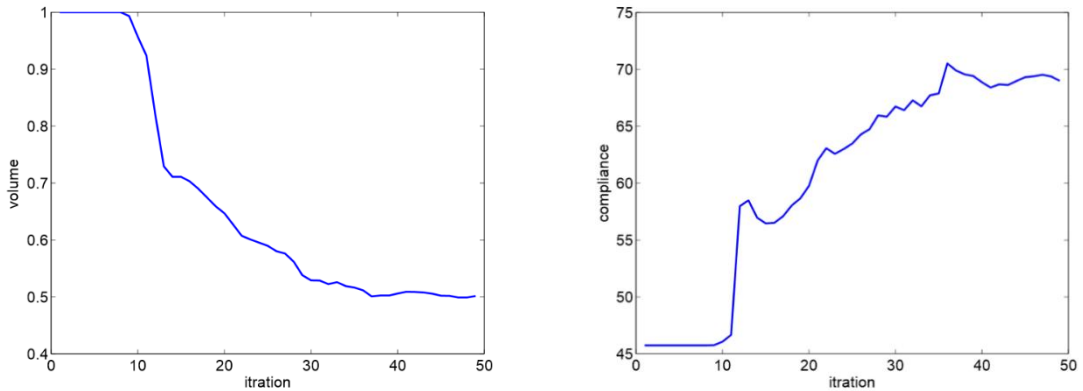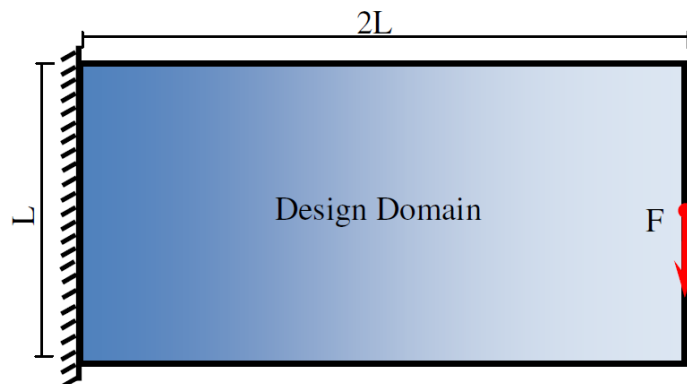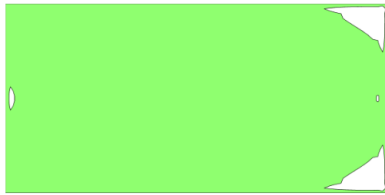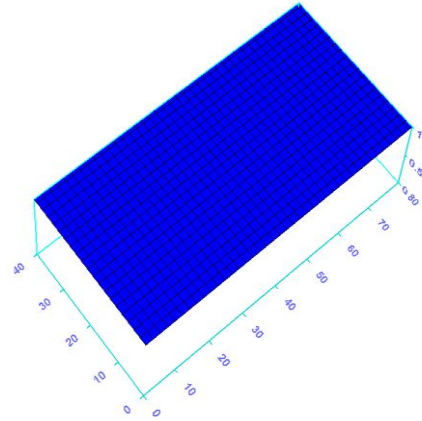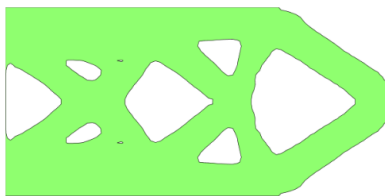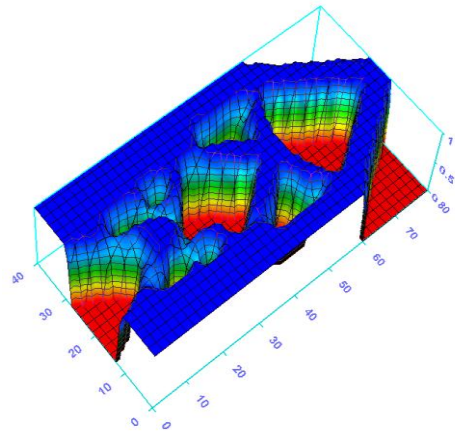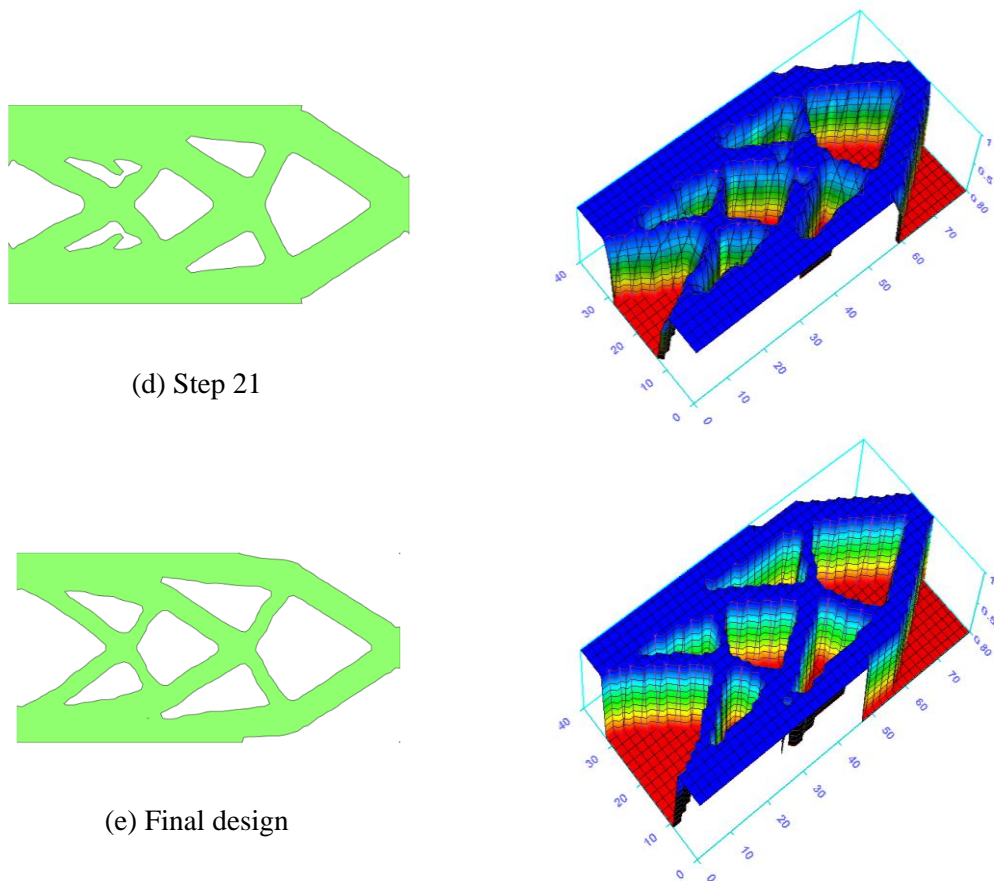
Figure 4. The evolution process of the compliance and the volume ratio

The evolution process of topology optimization is illustrated in Fig. 3. Fig. 4 depicts the history of convergence rate and the volume ratio for cantilever beam problem. The convergence rate and stability of strain energy verifies the efficiency of proposed method. Moreover, coupling phase field method with FEM results in smoother and thinner boundaries in comparison with other methods. The proposed method converges with fewer iterations comparing to AOS-MBO and MOS-MBO methods.

7.2 The cantilever beam 2

Fig. 5 shows the design domain of a cantilever beam. The boundary of the left side is fixed, and a vertical concentrated force $F = 1N$ is loaded in the middle of its right free side. The size of design domain is 80×40 with a squared mesh of size 1×1 and the volume fraction is 50%. This example is solved by PCLS-PFM scheme.

Figure 5. A cantilever beam

(a) Initial design

(b) Step 14

(c) Step 18

(d) Step 21

(e) Final design
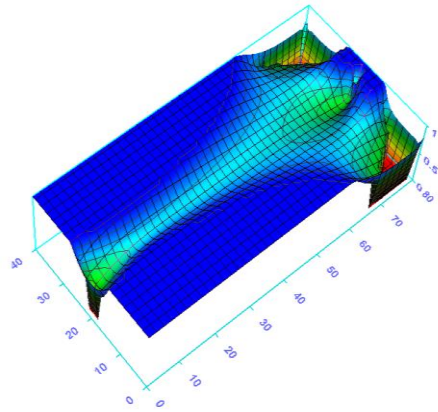
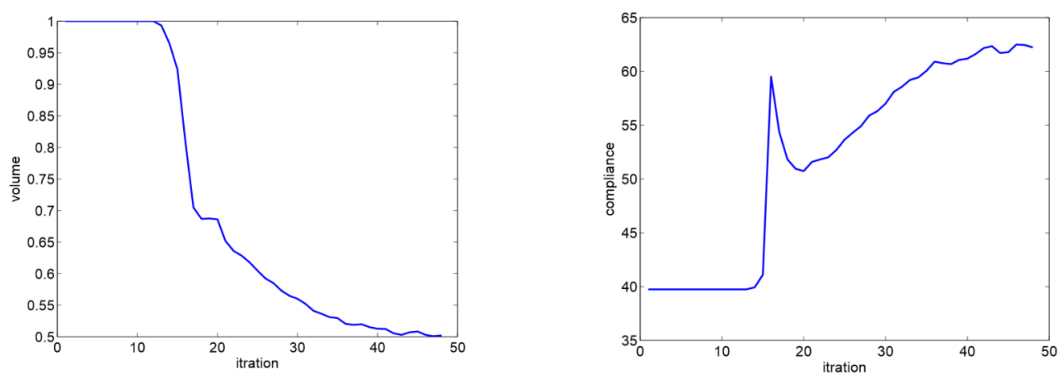Figure 6. The evolution process of optimal design and the level set function with Phase field method

Figure 7. The evolution process of the compliance and the volume ratio

7.3 The Messerschmitt–Bölkow–Blom beam

In the next example, we consider Messerschmitt–Bölkow–Blom (MBB) beam. The design

domain and the boundary condition of this type of structure are represented in Fig. 8. In this example the design domain is discretized with 30×120 squared elements of size 1×1. The volume fraction of the solid material is 50%. To find the optimal topology of this example, we apply the PCLS-PFM Scheme. Fig. 9 displays the evolution of an optimal topology of the MBB beam.
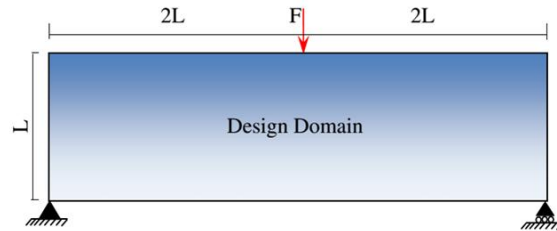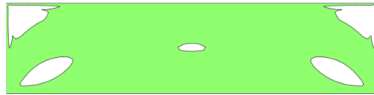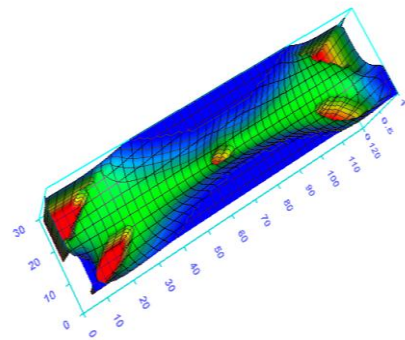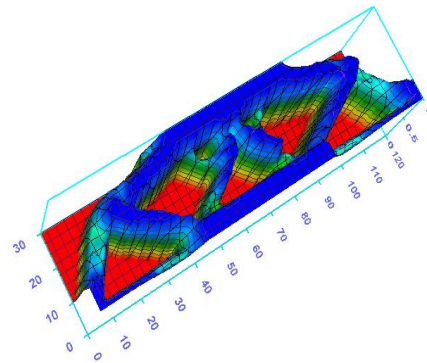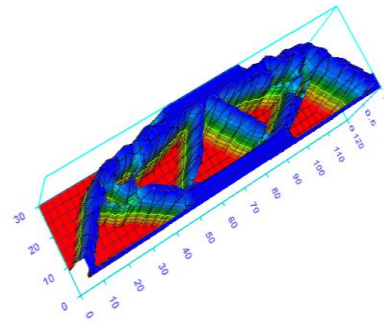

Figure 8. An MBB beam
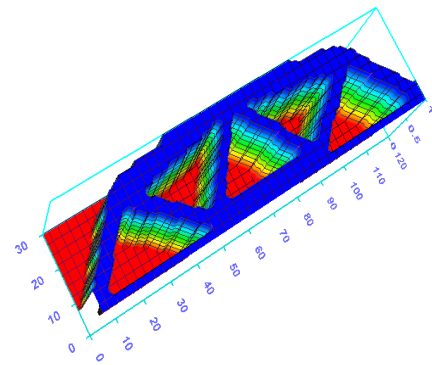

(b) Step 13

(c) Step 16

(d) Step 27

(e) Final design

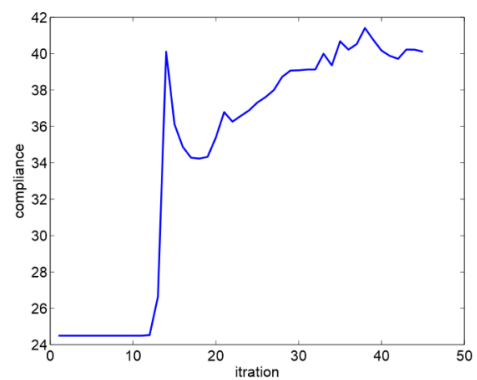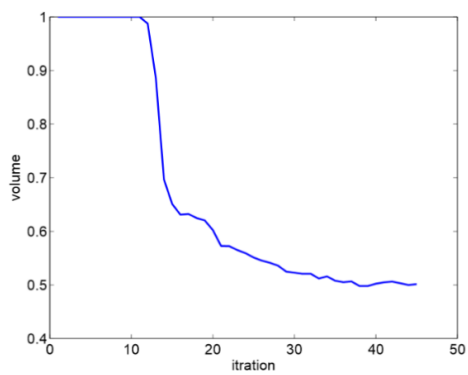Figure 9. The evolution process of optimal design and the level set function with Phase field method

Figure 10. The evolution process of the compliance and the volume ratio

8. CONCLUSIONS

In this study the improvement of piecewise constant level set method is investigated. The piecewise constant constraint is used to prevent the overlap or vacuum between two phases. The penalization method or Lagrange multiplier method can be used to apply the piecewise constant constraint. The penalization method is a stable method and feasible to use, however for accurate satisfaction of this constraint a small penalty coefficient should be considered. This will lead to instability of numerical solution process. In augmented Lagrange method there is no need to use a small value, but the number of iterations will increase. In the proposed method by omitting the piecewise constant constraint and addition of energy term based on phase field the convergence rate is increased. Moreover inner and outer boundaries are formed smoother and thinner. The resulting diffusion equation is solved through FEM which results in accurate solutions and better shapes.

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