6.1R

6.1R (Acceleration) The velocity in a certain flow field is given by the equation

$$\mathbf{V} = 3yz^2\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + y\hat{\mathbf{k}}$$

Determine the expressions for the three rectangular components of acceleration.

(ANS: $3xz^3 + 6y^2z$; $3yz^3 + xy$; xz)

From expression for velocity, $u = 3yz^2$, $v = xz$, and $w = y$.

Since

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

then

$$a_x = 0 + (3yz^2)(0) + (xz)(3z^2) + (y)(6yz)$$

$$= \underline{\underline{3xz^3 + 6y^2z}}$$

Similarly,

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

and

$$a_y = 0 + (3yz^2)(z) + (xz)(0) + (y)(x)$$

$$= \underline{\underline{3yz^3 + xy}}$$

Also,

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

so that

$$a_z = 0 + (3yz^2)(0) + (xz)(1) + (y)(0)$$

$$= \underline{\underline{xz}}$$

6.2R

6.2R (Vorticity) Determine an expression for the vorticity of the flow field described by

$$\mathbf{V} = x^2y\hat{\mathbf{i}} - xy^2\hat{\mathbf{j}}$$

Is the flow irrotational?

(ANS: $-(x^2 + y^2)\,\hat{\mathbf{k}}$; no)

The vorticity is twice the rotation vector:

$$\vec{\zeta} = 2\vec{\omega} = \nabla \times \vec{V} \qquad \text{(Eq. 6.17)}$$

From expression for velocity, $u = x^2y$, $v = -xy^2$, and $w = 0$, and with

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \text{(Eq. 6.13)}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad \text{(Eq. 6.14)}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \text{(Eq. 6.12)}$$

it follows that

$$\omega_x = 0, \qquad \omega_y = 0, \quad \text{and} \quad \omega_z = \frac{1}{2}\left(-y^2 - x^2\right)$$

Thus,

$$\vec{\zeta} = 2\left(\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}\right)$$

$$= 2\left[(0)\hat{i} + (0)\hat{j} + \frac{1}{2}(-y^2 - x^2)\hat{k}\right]$$

$$= \underline{\underline{-(x^2 + y^2)\,\hat{k}}}$$

Since $\vec{\zeta}$ is not zero everywhere, the flow is not irrotational. <u>No.</u>

6.3R

6.3R (Conservation of mass) For a certain incompressible, two-dimensional flow field the velocity component in the y direction is given by the equation

$$v = x^2 + 2xy$$

Determine the velocity component in the x direction so that the continuity equation is satisfied.
(ANS: $-x^2 + f(y)$)

To satisfy the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

Since

$$\frac{\partial v}{\partial y} = 2x$$

then from Eq. (1)

$$\frac{\partial u}{\partial x} = -2x \qquad (2)$$

Equation (2) can be integrated with respect to x to obtain

$$\int du = -\int 2x\, dx + f(y)$$

or

$$u = \underline{\underline{-x^2 + f(y)}}$$

where $f(y)$ is an undetermined function of y.

6.4R

6.4R (Conservation of mass) For a certain incompressible flow field it is suggested that the velocity components are given by the equations

$$u = x^2y \qquad v = 4y^3z \qquad w = 2z$$

Is this a physically possible flow field? Explain.

(ANS: No)

Any physically possible incompressible flow field must satisfy conseration of mass as expressed by the relationship

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (1)$$

For the velocity distribution given,

$$\frac{\partial u}{\partial x} = 2xy, \quad \frac{\partial v}{\partial y} = 12y^2z, \text{ and } \frac{\partial w}{\partial z} = 2$$

Substitution into Eq. (1) shows that

$$2xy + 12y^2z + 2 \neq 0 \text{ for all } x, y, z.$$

Thus, this is not a physically possible flow field. No.

6.5R

6.5R (Stream function) The velocity potential for a certain flow field is

$$\phi = 4xy$$

Determine the corresponding stream function.
(ANS: $2(y^2 - x^2) + C$)

For the given velocity potential,

$$u = \frac{\partial \phi}{\partial x} = 4y \quad \text{and} \quad v = \frac{\partial \phi}{\partial y} = 4x$$

From the definition of the stream function,

$$u = \frac{\partial \psi}{\partial y} = 4y \qquad (1)$$

Integrate Eq. (1) with respect to y to obtain

$$\int d\psi = \int 4y\, dy$$

or

$$\psi = 2y^2 + f_1(x) \qquad (2)$$
where $f_1(x)$ is an arbitrary function of x.

Similarly,

$$v = -\frac{\partial \psi}{\partial x} = 4x$$

and

$$\int d\psi = -\int 4x\, dx$$

or

$$\psi = -2x^2 + f_2(y) \qquad (3)$$
where $f_2(y)$ is an arbitrary function of y.

To satisfy both Eqs. (2) and (3) $f_1(x) = f_2(y)$ for all x and y.
Thus, $f_1 = f_2 =$ constant.

$$\psi = \underline{\underline{2(y^2 - x^2) + C}}$$

where C is a constant.

6.6R

6.6R (Velocity potential) A two-dimensional flow field is formed by adding a source at the origin of the coordinate system to the velocity potential

$$\phi = r^2 \cos 2\theta$$

Locate any stagnation points in the upper half of the coordinate plane ($0 \le \theta \le \pi$).
(ANS: $\theta_s = \pi/2$; $r_s = (m/4\pi)^{1/2}$)

$$\phi = \frac{m}{2\pi}\ln r + r^2\cos 2\theta\ , \text{ where } \phi_{source} = \frac{m}{2\pi}\ln r$$

Thus,

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -2r\sin 2\theta$$

and

$$v_r = \frac{\partial\phi}{\partial r} = \frac{m}{2\pi r} + 2r\cos 2\theta$$

Stagnation points will occur where $v_\theta = 0$, $v_r = 0$, for $0 \le \theta \le \pi$.
Thus,

$$0 = -2r_s \sin 2\theta_s \qquad (1)$$

$$0 = \frac{m}{2\pi r_s} + 2r_s\cos 2\theta_s \qquad (2)$$

Equation (1) is satisfied at $r_s = 0$ or $\theta_s = 0, \frac{\pi}{2}, \pi$.
From Eq. (2)

$$\cos 2\theta_s = -\frac{m}{4\pi r_s^2} \qquad (3)$$

and for the possible values of θ_s, only $\theta_s = \frac{\pi}{2}$ will satisfy Eq. (3). Recall that $m > 0$ for a source.
Thus,

$$r_s = \sqrt{\frac{m}{4\pi}}$$

Thus, the stagnation point is located at

$$\underline{\underline{\theta_s = \frac{\pi}{2}}}\ ,\quad \underline{\underline{r_s = \sqrt{\frac{m}{4\pi}}}}$$

6.7R

6.7R (Potential flow) The stream function for a two-dimensional, incompressible flow field is given by the equation

$$\psi = 2x - 2y$$

where the stream function has the units of ft^2/s with x and y in feet. **(a)** Sketch the streamlines for this flow field. Indicate the direction of flow along the streamlines. **(b)** Is this an irrotational flow field? **(c)** Determine the acceleration of a fluid particle at the point $x = 1$ ft, $y = 2$ ft.

(ANS: yes; no acceleration)

(a) Lines of constant ψ are streamlines. Thus, with $\psi = 2x - 2y$ the equation of a given streamline, ψ_1, (where ψ_1 is some constant) is of the form

$$\psi_1 = 2x - 2y$$

or

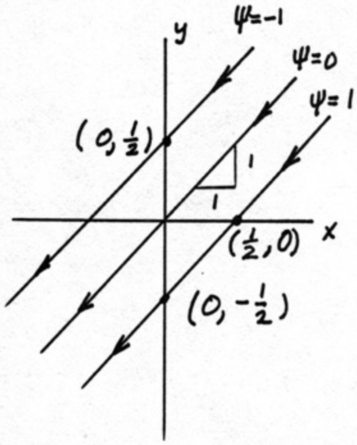
$$y = x - \frac{\psi_1}{2}$$

Thus, streamlines are straight lines as illustrated in the figure for three particular streamlines.

Since

$$u = \frac{\partial \psi}{\partial y} = -2 \qquad v = -\frac{\partial \psi}{\partial x} = -2$$

the direction of flow is as shown on the figure



(b) The flow field is irrotational if $\omega_z = 0$ where

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (Eq. 6.12)$$

For the stream function given

$$\frac{\partial v}{\partial x} = 0 \qquad \frac{\partial u}{\partial y} = 0$$

so that $\omega_z = 0$ and the flow field is irrotational. Yes.

(c) Since the velocity is constant throughout the flow field, the acceleration of all fluid particles is zero.

$$\vec{a} = \frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\nabla\vec{V} \equiv 0 \quad \text{since } \vec{V} = -2\hat{i} - 2\hat{j}$$

6.8R

6.8R (Inviscid flow) In a certain steady, incompressible, inviscid, two-dimensional flow field ($w = 0$, and all variables independent of z) the x component of velocity is given by the equation:

$$u = x^2 - y$$

Will the corresponding pressure gradient in the horizontal x direction be a function only of x, only of y, or of both x and y? Justify your answer.

(ANS: only of x)

Since the flow field must satisfy the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

and with $u = x^2 - y$ it follows that

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -2x$$

and therefore

$$v = -2xy + f_1(x)$$

For steady, two-dimensional flow of an inviscid fluid (with the x-axis horizontal so that $g_x = 0$) the x-component of the momentum equation is

$$-\frac{\partial p}{\partial x} = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) \qquad \text{(Eq. 6.51a)}$$

Thus, for the u and v given above

$$\frac{\partial p}{\partial x} = -\rho\left[(x^2 - y)(2x) + (-2xy + f_1(x))(-1)\right]$$

$$= \rho\left[f_1(x) - 2x^3\right] = F(x)$$

The pressure gradient $\frac{\partial p}{\partial x}$ is a function only of x.