of the liquid; that is, the pressure is the weight of a column of height ℓ and unit area. Therefore,

$$\begin{aligned} F_{\text{out}} &= c'u\sqrt{\frac{2\rho g \ell}{\rho}} \\ &= cu\sqrt{\ell} \end{aligned} \tag{2.29}$$

where c is a constant encompassing all constants in the equations.

From Equations 2.28 and 2.29, the state equation is

$$\dot{\ell} = -\frac{c}{A}u\sqrt{\ell} + \frac{F_{\text{in}}}{A}. \tag{2.30}$$

Specific values are $A = 1$ m^2, $c = 2.0$ m$^{3/2}$/s. With these,

$$\dot{\ell} = -2.0u\sqrt{\ell} + F_{\text{in}}. \tag{2.31}$$

The simulation conditions are as follows: $\ell(0) = 1$ m, $u(t) = .01$ m, $F_{\text{in}}(t) = 0$, $0 \leq t \leq 100$ s. These conditions correspond to the tank being emptied at constant valve opening. Figure 2.12 shows the behavior of the level (MATLAB command ode23). Note that the behavior is *not* exponential: the asymptotic value $\ell = 0$ is reached in finite time.

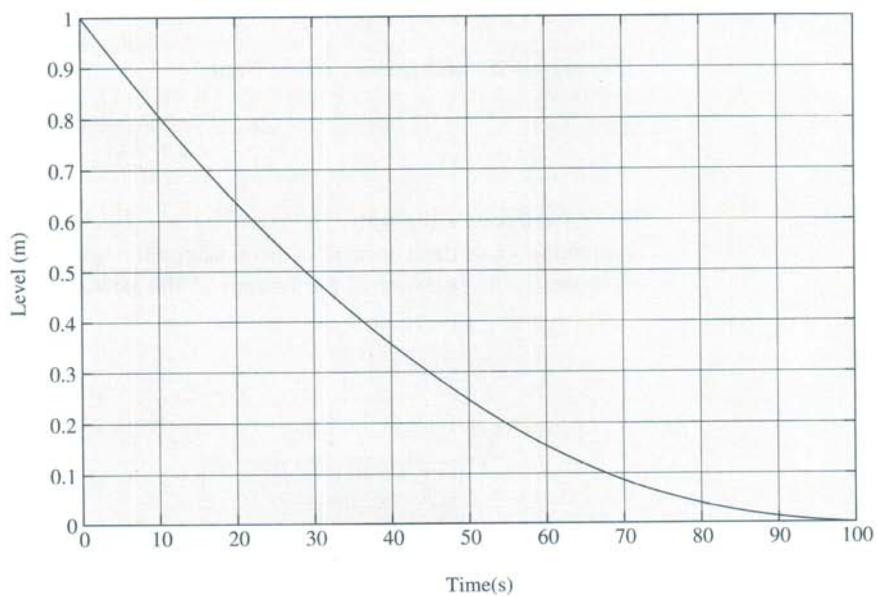

Figure 2.12 Response of level with zero in flow

2.4 MODELING WITH LAGRANGE'S EQUATIONS

Lagrange's equations constitute a well-known and useful technique for the analysis of mechanical systems [4,5]. To use Lagrange's equations, we define a set of *generalized coordinates*, that is, a set of positions and angles that completely describe the motion of the system. These coordinates must be independent; that is, motion obtained by arbitrary specification of coordinate time history must be mechanically possible.

The kinetic energy of the system is a function of the generalized coordinates q_i and their derivatives, and is written as $T(\mathbf{q}, \dot{\mathbf{q}})$. The potential energy is a function of the q_i and is written as $V(\mathbf{q})$.

The *Lagrangian* L is defined as

$$L = T - V. \tag{2.32}$$

To write Lagrange's equations, we need to define the *generalized forces*, F_i. We do this by computing the work done by all nonconservative forces when q_i is changed to $q_i + dq_i$ with all other coordinates held fixed. For infinitesimal dq_i, the work is proportional to dq_i, and the proportionality factor is F_i.

Lagrange's equations are as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i, \qquad i = 1, 2, \ldots, n. \tag{2.33}$$

Example 2.5 **(Pendulum on a Cart)**

Description: An inverted pendulum of mass m and length ℓ moves in the vertical plane, about a horizontal axis fixed on a cart. The cart, of mass M, moves horizontally in one dimension, under the influence of a force F. (See Fig. 2.13). The pendulum rod is assumed to have zero mass. There is no friction in the system. The force F is to be manipulated to keep the pendulum vertical.

Inputs and Outputs: The input is the force F, and the outputs are the angle θ and the distance x.

Objective: Write the equations, and simulate under given conditions.

Solution The generalized coordinates are x and θ. The velocity of m has two components, one due to the motion of the cart and the other due to the angular motion of the pendulum. The velocity of the cart is $\dot{x}$ in the horizontal direction.

The horizontal position of the mass m is $x + \ell \sin\theta$, and its vertical position is $\ell\cos\theta$. Therefore, the total kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[\left\{\frac{d}{dt}(x + \ell\sin\theta)\right\}^2 + \left\{\frac{d}{dt}(\ell\cos\theta)\right\}^2\right] \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[(\dot{x} + \ell\dot{\theta}\cos\theta)^2 + (-\ell\dot{\theta}\sin\theta)^2]. \end{aligned}$$

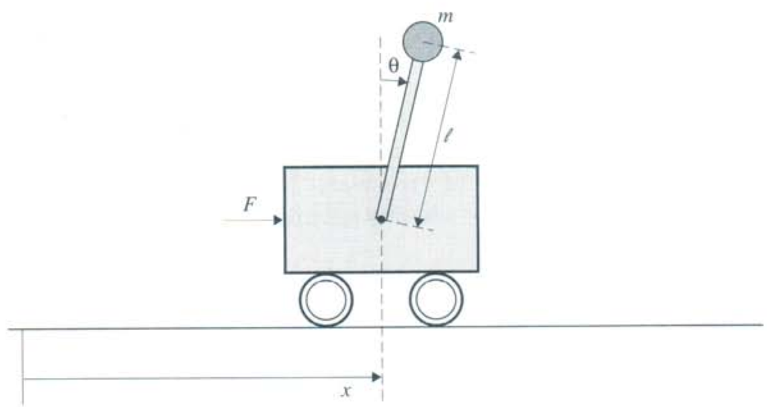

Figure 2.13 Pendulum on a cart

The potential energy of m varies with height. If V_0 is the potential energy of m for $\theta = 90^\circ$, then

$$V = V_0 + mg\ell \cos\theta.$$

Thus,

$$L = \frac{1}{2}M(\dot{x})^2 + \frac{1}{2}m[(\dot{x} + \ell\dot{\theta}\cos\theta)^2 + (\ell\dot{\theta}\sin\theta)^2] - V_0 - mg\ell\cos\theta.$$

The only nonconservative force is F. If x is held fixed and θ is changed to $\theta + d\theta$, F does no work: the generalized force associated with θ is zero. If θ is held fixed and x changes to $x + dx$, the work done is $F dx$; therefore, F is the generalized force associated with x.

We may now write Lagrange's equations:

$$\begin{aligned}\frac{\partial L}{\partial \dot{x}} &= M\dot{x} + m(\dot{x} + \ell\dot{\theta}\cos\theta)\\ \frac{\partial L}{\partial x} &= 0\\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= M\ddot{x} + m\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell(\dot{\theta})^2\sin\theta.\end{aligned}$$

The equation related to x is

$$(M + m)\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell(\dot{\theta})^2\sin\theta = F. \qquad \textbf{(2.34)}$$

For the θ equations,

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}} &= m(\dot{x} + \ell\dot{\theta}\cos\theta)\ell\cos\theta + m\ell^2\dot{\theta}\sin^2\theta \\
&= m\ell\dot{x}\cos\theta + m\ell^2\dot{\theta} \\
\frac{\partial L}{\partial \theta} &= -m(\dot{x} + \ell\dot{\theta}\cos\theta)\ell\dot{\theta}\sin\theta + m\ell^2(\dot{\theta})^2\sin\theta\cos\theta + mg\ell\sin\theta \\
&= -m\ell\dot{\theta}\dot{x}\sin\theta + mg\ell\sin\theta \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= m\ell\ddot{x}\cos\theta - m\ell\dot{\theta}\dot{x}\sin\theta + m\ell^2\ddot{\theta}.
\end{aligned}
$$

The equation pertaining to θ is

$$m\ell\ddot{x}\cos\theta - m\ell\dot{\theta}\dot{x}\sin\theta + m\ell^2\ddot{\theta} + m\ell\dot{\theta}\dot{x}\sin\theta - mg\ell\sin\theta = 0$$

or

$$\ddot{x}\cos\theta + \ell\ddot{\theta} - g\sin\theta = 0. \tag{2.35}$$

Equations 2.34 and 2.35 are not state equations.

Define $v = \dot{x}$ and $\omega = \dot{\theta}$, and write Equations 2.34 and 2.35 as

$$\begin{bmatrix} M+m & m\ell\cos\theta \\ \cos\theta & \ell \end{bmatrix}\begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} F + m\ell\omega^2\sin\theta \\ g\sin\theta \end{bmatrix}.$$

Solving for $\dot{v}$ and $\dot{\omega}$ yields

$$\dot{v} = \frac{F + m\ell\omega^2\sin\theta - mg\sin\theta\cos\theta}{M + m(1-\cos^2\theta)} \tag{2.36}$$

$$\dot{\omega} = \frac{-F\cos\theta - m\ell\omega^2\sin\theta\cos\theta + (M+m)g\sin\theta}{\ell[M + m(1-\cos^2\theta)]}. \tag{2.37}$$

Append the definitions

$$\dot{x} = v \tag{2.38}$$

$$\dot{\theta} = \omega. \tag{2.39}$$

Equations 2.36 to 2.39 are the four state equations. Specific values are $\ell = 1$ m and $M = m = 1$ kg. The state equations are

$$
\begin{aligned}
\dot{x} &= v \\
\dot{\theta} &= \omega \\
\dot{v} &= \frac{F + \omega^2\sin\theta - 9.8\sin\theta\cos\theta}{2 - \cos^2\theta} \\
\dot{\omega} &= \frac{-F\cos\theta - \omega^2\sin\theta\cos\theta + 19.6\sin\theta}{2 - \cos^2\theta}.
\end{aligned} \tag{2.40}
$$

The simulation conditions are as follows: $x(0) = v(0) = \omega(0) = 0, \theta(0) = 0.1$ rad, $F(t) = 0, 0 \leq t \leq 1$ s. Figure 2.14 shows the results (MATLAB command ode23). This system is seen to be unstable. The pendulum falls to the right ($\theta > 0$) while the cart goes to the left.

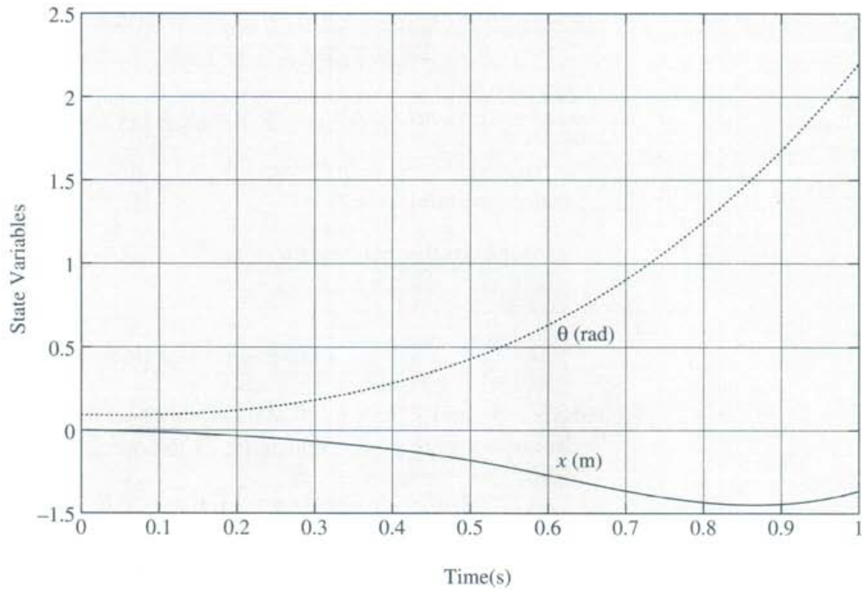

Figure 2.14 Pendulum angle and cart distance from a nonzero initial state

2.5 LINEARIZATION

The task of a control system is often to maintain given constant operating conditions—for example, constant speed, level, position, or basis weight. To achieve this objective, we use a two-step procedure:

1. Select a dc steady state that corresponds to desired constant values of $\mathbf{u}$ and/or $\mathbf{y}$.
2. Design a control strategy to generate increments in the control in response to deviations from the dc steady state.

To do this, we need to study (i) the dc steady state of a system and (ii) the model that relates the deviations from steady state, i.e., the *small-signal*, or *incremental*, model.

We begin with

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{2.41}$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}). \tag{2.42}$$

Note that the functions $\mathbf{f}$ and $\mathbf{h}$ are not explicitly functions of t, so the system is time-invariant.

For constant $\mathbf{u} = \mathbf{u}^*$, $\mathbf{x}^*$ is an *equilibrium state* if $\mathbf{f}(\mathbf{x}^*, \mathbf{u}^*) = \mathbf{0}$. We shall use the symbol $\mathbf{0}$ to denote a vector whose elements are all 0. If $\mathbf{x} = \mathbf{x}^*$ and $\mathbf{u} = \mathbf{u}^*$, then $\dot{\mathbf{x}} = \mathbf{0}$ and the state remains at $\mathbf{x}^*$; i.e., $\mathbf{x}^*$ is an equilibrium point with $\mathbf{u} = \mathbf{u}^*$.

The output corresponding to an equilibrium state $\mathbf{x}^*$ is $\mathbf{y}^* = \mathbf{h}(\mathbf{x}^*, \mathbf{u}^*)$. Therefore, the dc steady-state quantities satisfy

$$\begin{aligned} \mathbf{f}(\mathbf{x}^*, \mathbf{u}^*) &= \mathbf{0} \\ \mathbf{h}(\mathbf{x}^*, \mathbf{u}^*) &= \mathbf{y}^*. \end{aligned} \tag{2.43}$$

A dc steady state is defined by choosing some of the variables in Equation (2.43) and solving for the others. There is no guarantee that a solution will exist, or that it will be unique. With n states, r inputs, and m outputs, Equation 2.43 represents $n + m$ nonlinear equations with $n + m + r$ variables. In most cases, it will not be possible to predetermine more than r of those variables. For example, it will not usually be possible to set 2 outputs ($m = 2$) at arbitrary values if the system has only one input ($r = 1$).

The next step is to write equations for incremental variables, i.e., for deviations from equilibrium. Let

$$\mathbf{x}(t) = \mathbf{x}^* + \Delta\mathbf{x}(t), \qquad \mathbf{u}(t) = \mathbf{u}^* + \Delta\mathbf{u}(t), \qquad \mathbf{y}(t) = \mathbf{y}^* + \Delta\mathbf{y}(t).$$

Because $\dot{\mathbf{x}}^* = \mathbf{0}$, substitution in Equations 2.41 and 2.42 yields

$$\Delta\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}^* + \Delta\mathbf{x}, \mathbf{u}^* + \Delta\mathbf{u}) \tag{2.44}$$

$$\Delta\mathbf{y} = \mathbf{h}(\mathbf{x}^* + \Delta\mathbf{x}, \mathbf{u}^* + \Delta\mathbf{u}) - \mathbf{y}^*. \tag{2.45}$$

Expanding the components of $\mathbf{f}$ in a Taylor series, we obtain

$$\begin{aligned} f_i(\mathbf{x}^* + \Delta\mathbf{x}, \mathbf{u}^* + \Delta\mathbf{u}) = f_i(\mathbf{x}^*, \mathbf{u}^*) &+ \left.\frac{\partial f_i}{\partial x_1}\right|_* \Delta x_1 + \left.\frac{\partial f_i}{\partial x_2}\right|_* \Delta x_2 + \cdots \\ &+ \left.\frac{\partial f_i}{\partial x_m}\right|_* \Delta x_n + \left.\frac{\partial f_i}{\partial u_1}\right|_* \Delta u_1 + \cdots + \left.\frac{\partial f_i}{\partial u_r}\right|_* \Delta u_r \\ &+ \text{higher-order terms in } \Delta x, \Delta u. \end{aligned} \tag{2.46}$$

Here, the notation "$\Big|_*$" means "evaluated at $\mathbf{x}^*, \mathbf{u}^*$." At this point, it is assumed that the Δx's and Δu's are sufficiently small to justify neglecting the higher-order terms. If the control system to be designed works at all well, that assumption should be satisfied.

Without the higher-order terms, and with $\mathbf{f}(\mathbf{x}^*, \mathbf{u}^*) = \mathbf{0}$, the RHS of Equation 2.46 is the ith member of a set of n equations, written in matrix form as

$$\mathbf{f}(\mathbf{x}^* + \Delta\mathbf{x},\ \mathbf{u}^* + \Delta\mathbf{u}) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_* \Delta\mathbf{x} + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_* \Delta\mathbf{u} \tag{2.47}$$

where

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

is the Jacobian of **f** with respect to **x**, with a similar definition for $\frac{\partial \mathbf{f}}{\partial \mathbf{u}}$, the Jacobian with respect to **u**. Thus, Equation 2.44 becomes approximately

$$\Delta\dot{\mathbf{x}} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_* \Delta\mathbf{x} + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_* \Delta\mathbf{u}. \tag{2.48}$$

As for Equation 2.45, since $\mathbf{y}^* = \mathbf{h}(\mathbf{x}^*, \mathbf{u}^*)$, we have

$$\Delta\mathbf{y} = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_* \Delta\mathbf{x} + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right|_* \Delta\mathbf{u} \tag{2.49}$$

for small $\Delta\mathbf{x}$, $\Delta\mathbf{u}$.

Note that the Jacobians in Equations 2.48 and 2.49 are constant matrices, because they are evaluated at specific values $\mathbf{x}^*$ and $\mathbf{u}^*$. Note also that the right-hand sides of those equations are linear functions of $\Delta\mathbf{x}$ and $\Delta\mathbf{u}$, so the incremental system is *linear* and *time-invariant*.

It is also possible to linearize about a *trajectory*—a nominal set of time functions, $\mathbf{x}^*(t)$ and $\mathbf{u}^*(t)$, that satisfy the state equations. An example would be a robotic manipulator following a preset path. In such a case, the linearized system is *time-varying* (see Problem 2.21).

If some of the inputs are disturbances, it is often desirable to separate them from the control inputs. The linearized equations become

$$\Delta\dot{\mathbf{x}} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_* \Delta\mathbf{x} + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_* \Delta\mathbf{u} + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{w}}\right|_* \Delta\mathbf{w} \tag{2.50}$$

$$\Delta\mathbf{y} = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_* \Delta\mathbf{x} + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right|_* \Delta\mathbf{u} + \left.\frac{\partial \mathbf{h}}{\partial \mathbf{w}}\right|_* \Delta\mathbf{w} \tag{2.51}$$

where $\mathbf{w}$ is the vector of disturbance inputs.

If the original system is linear and time-invariant, it is represented by equations of the form

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} + F\mathbf{w} \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u} + G\mathbf{w}. \end{aligned} \tag{2.52}$$

The equilibrium point satisfies

$$\begin{aligned} \mathbf{0} &= A\mathbf{x}^* + B\mathbf{u}^* + F\mathbf{w}^* \\ \mathbf{y}^* &= C\mathbf{x}^* + D\mathbf{u}^* + G\mathbf{w}^*. \end{aligned} \tag{2.53}$$

If $\mathbf{u}^*$ and $\mathbf{w}^*$ are given, a unique solution $\mathbf{x}^*$ (hence $\mathbf{y}^*$) always exists if A is nonsingular. If A is singular, there are multiple solutions if the vector $B\mathbf{u}^* + F\mathbf{w}^*$ is in the *range space* of A, i.e., can be constructed by a linear combination of the columns of A; if that is not the case, there is no solution.

If $\mathbf{y}^*$ and $\mathbf{w}^*$ are given and we wish to solve for $\mathbf{x}^*$ and $\mathbf{u}^*$, it is useful to write Equation 2.53 as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{x}^* \\ \mathbf{u}^* \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{y}^* \end{bmatrix} - \begin{bmatrix} F \\ G \end{bmatrix} \mathbf{w}^*. \tag{2.54}$$

If $m = r$ (equal number of inputs and outputs), then the matrix on the left-hand side (LHS) of Equation 2.54 is square, and a unique solution exists if that matrix is nonsingular. If $r > m$ (more inputs than outputs), and if the matrix has maximal rank $n + m$, there exist multiple solutions to Equation 2.54. Finally, if $r < m$ (fewer inputs than outputs) and the matrix has maximal rank $n + r$, there is a (unique) solution only in the special case where $\mathbf{y}^*$ and $\mathbf{w}^*$ are such that the RHS of Equation 2.54 in the range space of the matrix $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.

As for the incremental system, with

$$\mathbf{x} = \mathbf{x}^* + \Delta\mathbf{x}, \qquad \mathbf{u} = \mathbf{u}^* + \Delta\mathbf{u}, \qquad \mathbf{y} = \mathbf{y}^* + \Delta\mathbf{y}, \qquad \mathbf{w} = \mathbf{w}^* + \Delta\mathbf{w}$$

Equations 2.52 become

$$\Delta\dot{\mathbf{x}} = A\mathbf{x}^* + B\mathbf{u}^* + F\mathbf{w}^* + A\,\Delta\mathbf{x} + B\,\Delta\mathbf{u} + F\Delta\mathbf{w}$$
$$\mathbf{y}^* + \Delta\,\mathbf{y} = C\mathbf{x}^* + D\mathbf{u}^* + G\mathbf{w}^* + C\,\Delta\mathbf{x} + D\,\Delta\mathbf{u} + G\,\Delta\mathbf{w}$$

which, in view of Equation 2.53, yields

$$\begin{aligned}\Delta\dot{\mathbf{x}} &= A\Delta\mathbf{x} + B\Delta\mathbf{u} + F\Delta\mathbf{w}\\ \Delta\mathbf{y} &= C\Delta\mathbf{x} + D\Delta\mathbf{u} + G\Delta\mathbf{w}.\end{aligned} \tag{2.55}$$

Equation 2.55 expresses the fact that a linear system is its own incremental system, and therefore no extra work is needed to obtain the incremental system in that case.

Example 2.6 (dc Servo)

For the servomechanism of Example 2.1, calculate the constant equilibrium point for $T_L = 0$ and $\theta^* = \theta_d$. Give the incremental model.

Solution From Equation 2.17 and the first of the two output equations in Equation 2.18, application of Equation 2.53 yields

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & \dfrac{NK_m}{J_e}\\ 0 & \dfrac{-NK_m}{L} & -\dfrac{R}{L}\end{bmatrix}\begin{bmatrix}\theta^*\\ \omega^*\\ i^*\end{bmatrix} + \begin{bmatrix}0\\0\\ \dfrac{1}{L}\end{bmatrix} v^*$$

$$\theta_d = [1 \quad 0 \quad 0]\begin{bmatrix}\theta^*\\ \omega^*\\ i^*\end{bmatrix}.$$

It follows easily that $\omega^* = i^* = v^* = 0$.

The incremental variables are

$$\Delta\theta = \theta - \theta_d, \qquad \Delta\omega = \omega - \omega^* = \omega, \qquad \Delta i = i - i^* = i, \qquad \Delta v = v - v^* = v.$$

Following Equation 2.55, the incremental model is

$$\frac{d}{dt}\begin{bmatrix}\Delta\theta\\ \omega\\ i\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & \dfrac{NK_m}{J_e}\\ 0 & \dfrac{-NK_m}{L} & \dfrac{-R}{L}\end{bmatrix}\begin{bmatrix}\Delta\theta\\ \omega\\ i\end{bmatrix} + \begin{bmatrix}0\\0\\ \dfrac{1}{L}\end{bmatrix} v$$

$$\begin{bmatrix}\Delta\theta\\ \omega\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\Delta\theta\\ \omega\\ i\end{bmatrix}.$$