Structural Dynamics

A Graduate Course in Aerospace Engineering

By: **H. Ahmadian**

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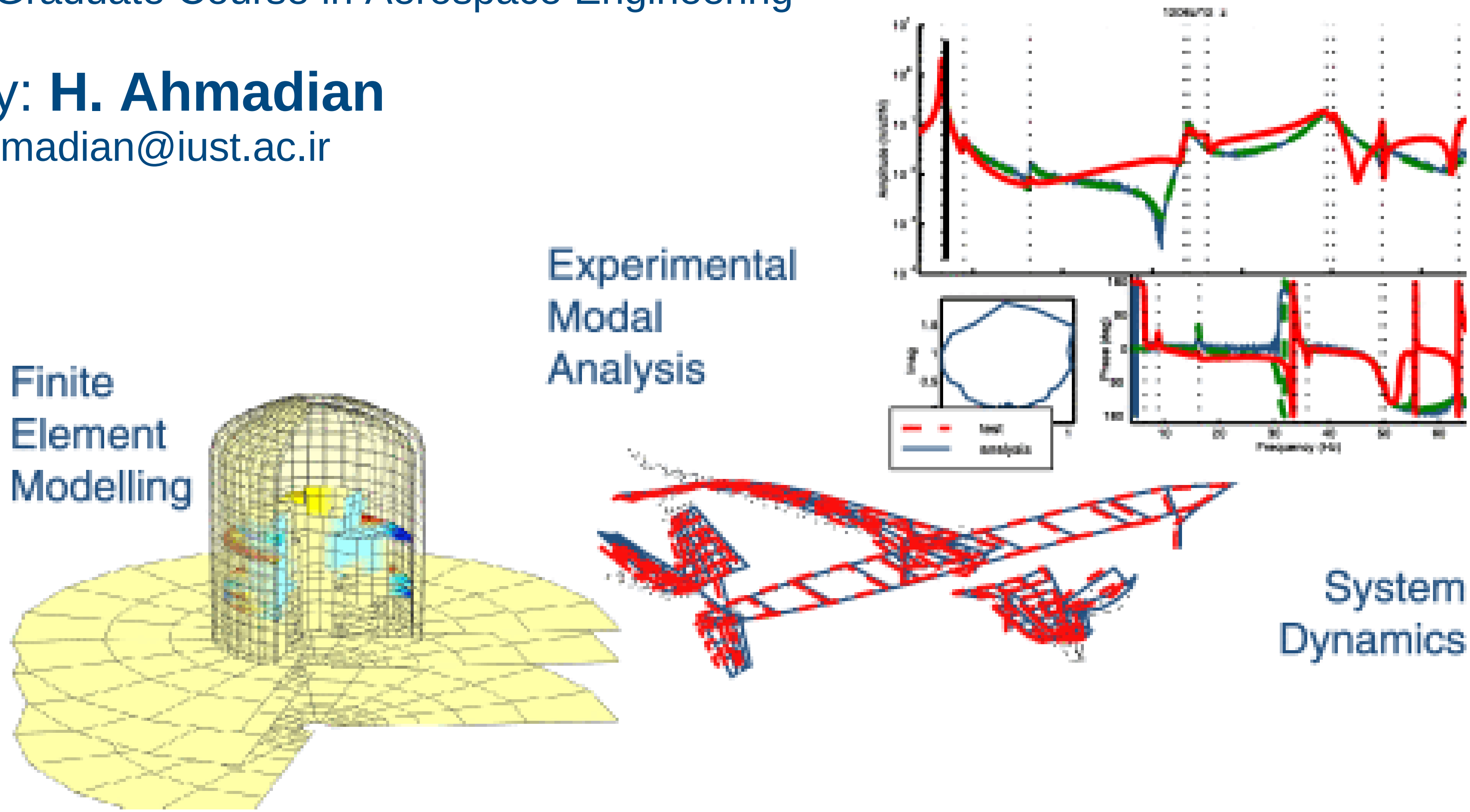


School of Mechanical Engineering

Iran University of Science and Technology

The Science and Art of Structural Dynamics

What do all the followings have in common?

> A sport-utility vehicle traveling off-road,
> An airplane flying near a thunderstorm,
> An offshore oil platform in rough seas, and
> An office tower during an earthquake.

All these structures are subjected to ***dynamic loading***, that is, to time-varying loading.

Safety, performance, and ***reliability*** of structures led to the need for extensive analysis and testing to determine their ***response to dynamic loading***.

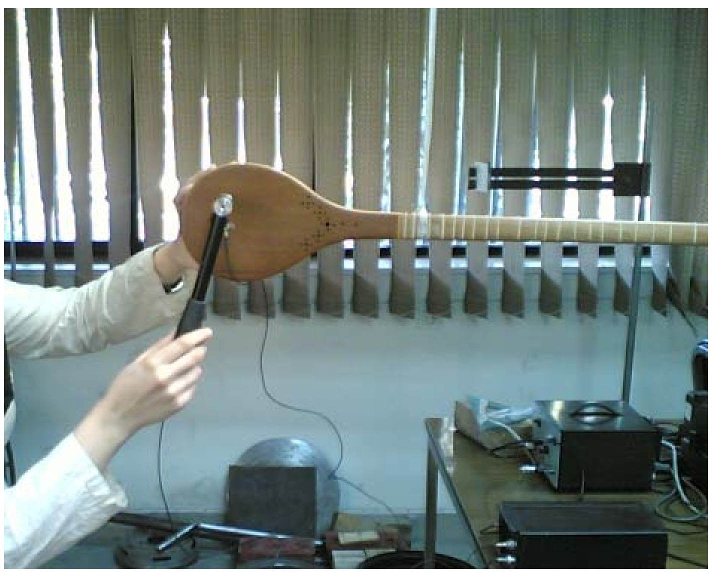
The Science and Art of Structural Dynamics

Although the topic of this course, as indicated by its title, is ***structural dynamics***, some courses with the word ***vibrations*** in their title discuss essentially the same subject matter.

Powerful computer programs are invariably used to implement the modeling, analysis, and testing tasks that are discussed in this course,

The Science and Art of Structural Dynamics

The application is in aerospace engineering, civil engineering, mechanical engineering, electrical engineering, or even in sports or music.



INTRODUCTION TO STRUCTURAL DYNAMICS

By studying the principles and mathematical formulations discussed in this course you will begin to understand the ***science of structural dynamics analysis***.

However, structural dynamicists must also master the *art* of ***creating mathematical models of structures***, and in many cases they must also perform ***dynamic tests***.

INTRODUCTION TO STRUCTURAL DYNAMICS

A ***dynamic -load*** is one whose magnitude, direction, or point of application **varies with time**.

The ***resulting*** time-varying displacements and stresses constitute the ***dynamic response***.

If the ***loading is a known*** function of time, the analysis of a given structural system to a known loading is called a ***deterministic analysis***.

If the ***time history*** of the loading is not known completely but only in a ***statistical sense***, the loading is said to be ***random***.

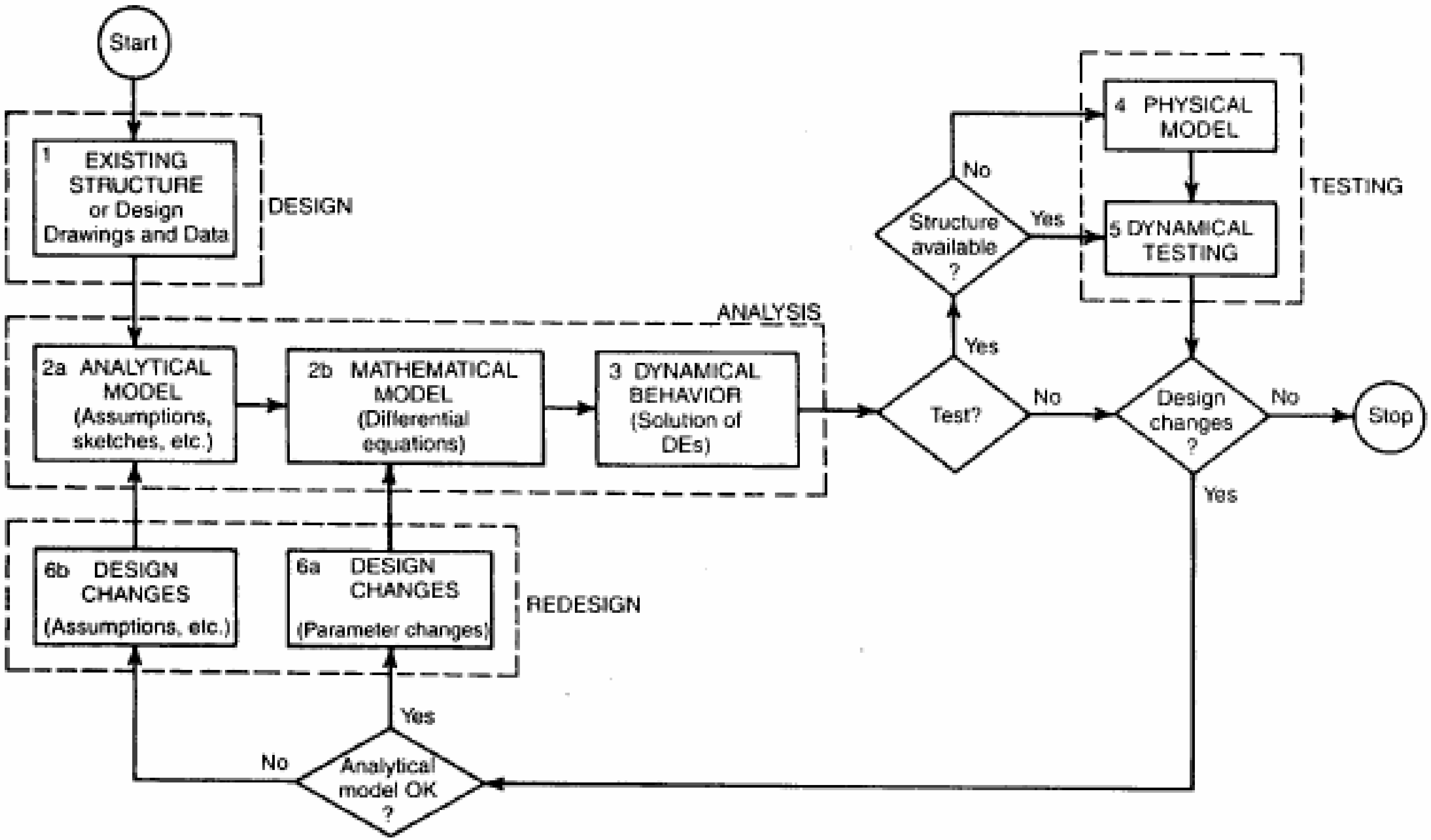
INTRODUCTION TO STRUCTURAL DYNAMICS

A structural dynamics problem differs from the corresponding static problem in two important respects:

- The time-varying nature of the *excitation*.
- The role played by *acceleration*.

If the inertia force contributes significantly to the deflection of the structure and the internal stresses in the structure, a dynamical investigation is required.

Steps in a Dynamical Investigation



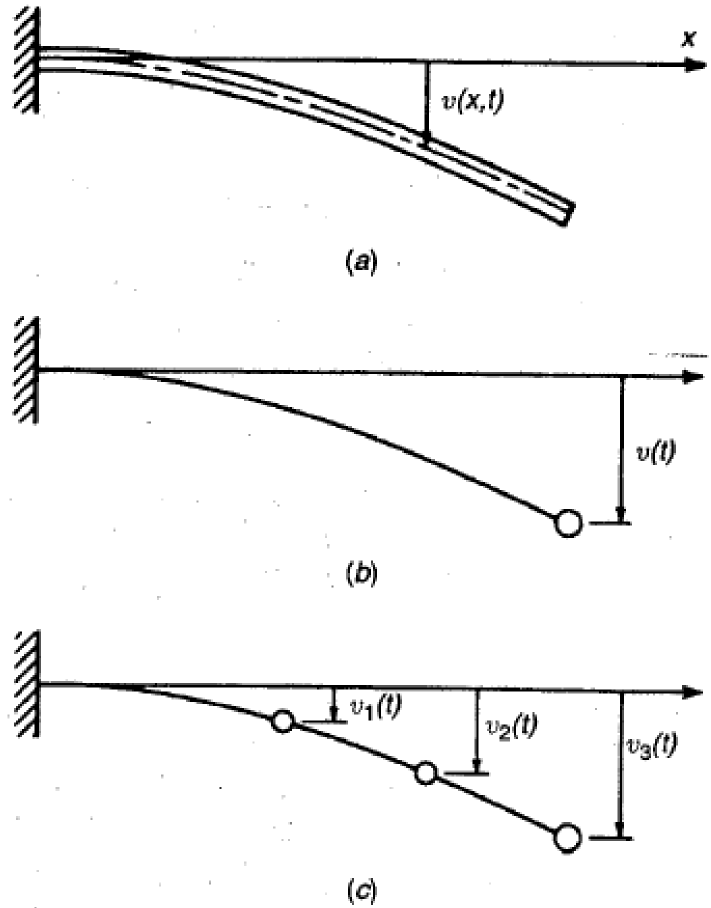
MODELING OF STRUCTURAL COMPONENTS AND SYSTEMS

Perhaps the most demanding step in any dynamical analysis is the creation of a *mathematical model* of the structure.

This *analytical model* consists of:

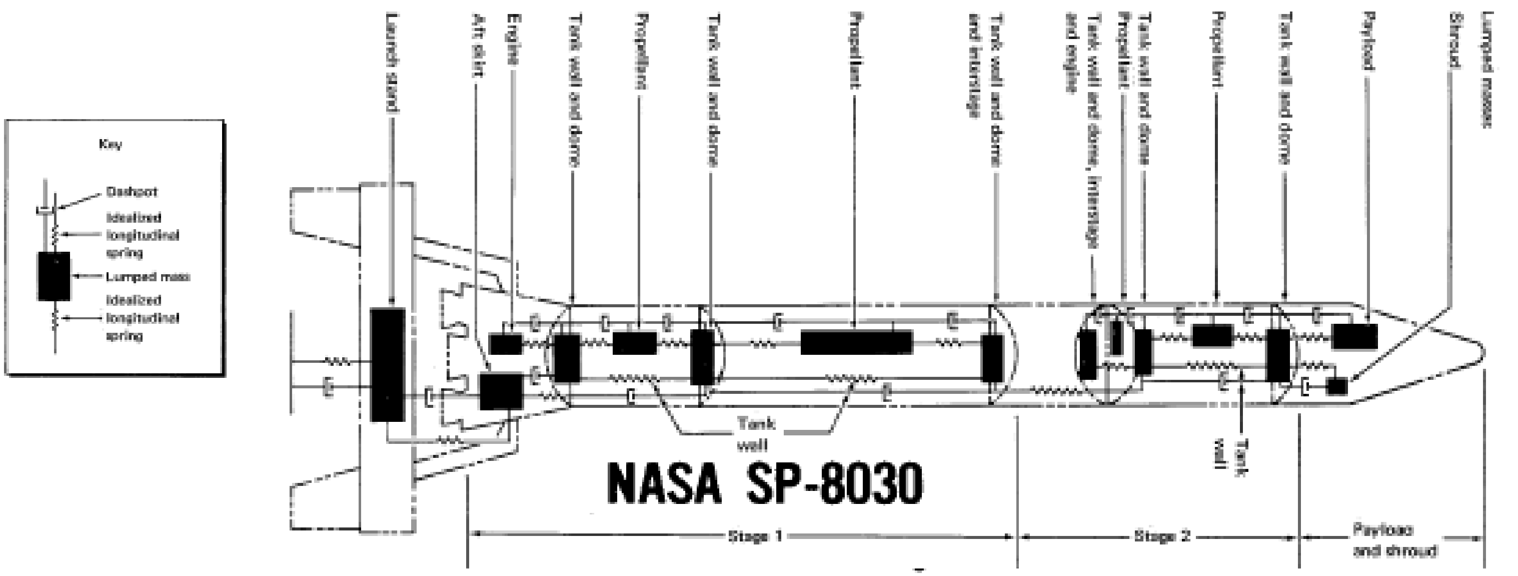
1. A list of the simplifying assumptions made in reducing the real system to the analytical model
2. Drawings that depict the analytical model
3. A list of the design parameters (i.e., sizes, materials, etc.)

Analytical models fall into two basic categories: ***continuous models*** and ***discrete parameter models***.



MODELING OF STRUCTURAL COMPONENTS AND SYSTEMS

To create a useful analytical model, you must have clearly in mind the intended use of the analytical model, that is; the types of behavior of the real system that the model is supposed to represent faithfully.



Simplified longitudinal dynamic model of space vehicle held to launch stand

MODELING OF STRUCTURAL COMPONENTS AND SYSTEMS

The complexity of the analytical model is determined by:

(1) the types and detail of behavior that it must represent,

(2) the computational analysis capability available (hardware and software), and

(3) the time and expense allowable.

MODELING OF STRUCTURAL COMPONENTS AND SYSTEMS

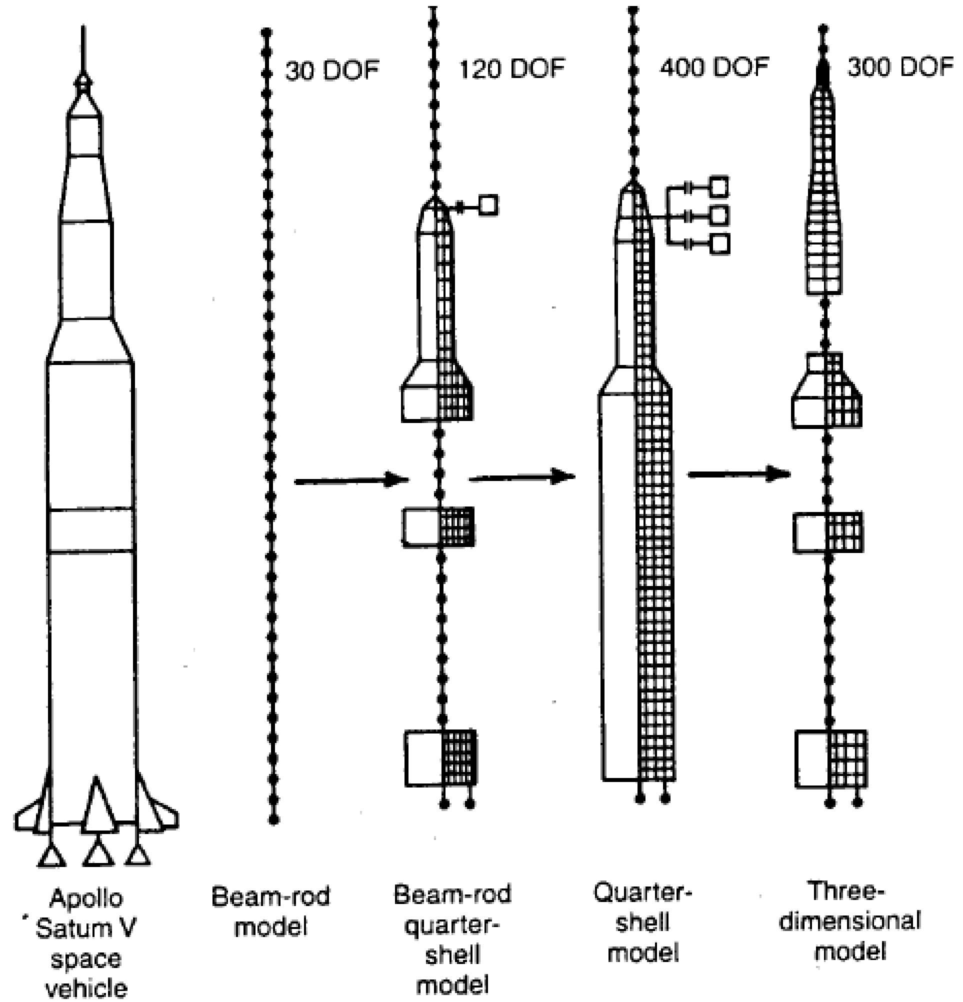
30-DOF beam-rod model

- preliminary studies and to determine full-scale testing requirements.

The 300-DOF model

- a more accurate description of motion at the flight sensor locations.

Simplicity of the analytical model is very desirable as long as the model is adequate to represent the necessary behavior.



MODELING OF STRUCTURAL COMPONENTS AND SYSTEMS

Once you have ***created an analytical model***, you can ***apply physical laws***' to obtain the ***differential equation(s)*** of motion that describe, in mathematical language, the analytical model.

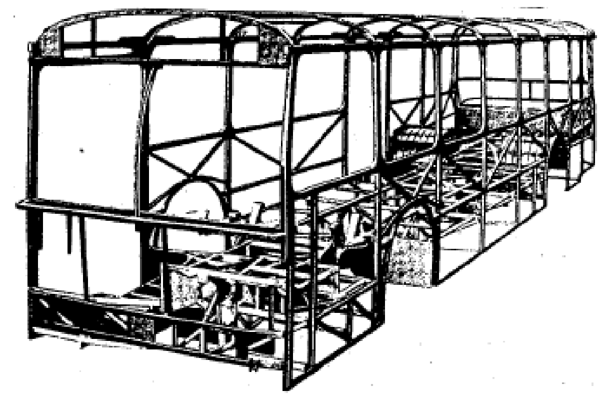
- A continuous model leads to partial differential equations, whereas a
- discrete-parameter model leads to ordinary differential equations.

MODELING OF STRUCTURAL COMPONENTS AND SYSTEMS

In using a finite element computer program, your major modeling task will be to simplify the system and provide input data on dimensions, material properties, loads, and so on.

This is where the "art" of structural dynamics comes into play.

On the other hand, actual creation and solution of the differential equations is done by the computer program.



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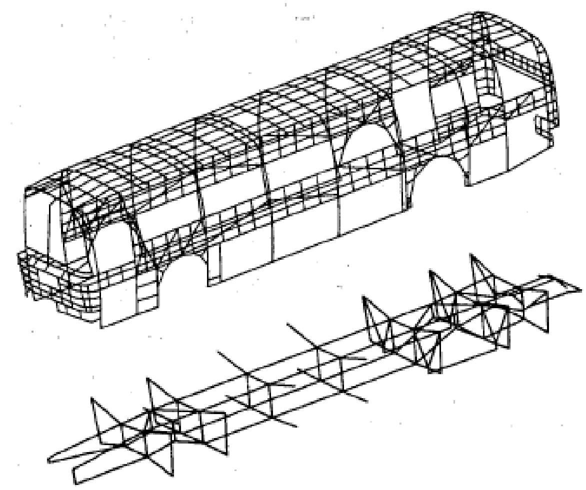


VIBRATION TESTING OF STRUCTURES

A primary purpose of dynamical testing is to:

- confirm a mathematical model and,
- to obtain important information on loads, on damping, and on other parameters that may be required in the dynamical analysis.

In some instances these tests are conducted on ***reduced-scale*** *physical models*

In other cases, when a ***full-scale*** structure is available, the tests may be conducted on it.

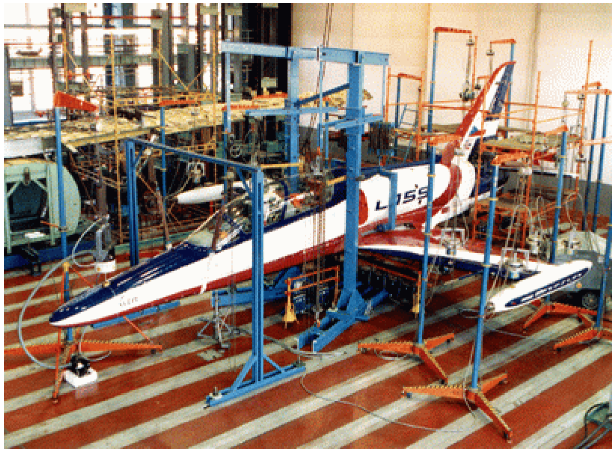
VIBRATION TESTING OF STRUCTURES

Aerospace vehicles must be subjected to extensive static and dynamic testing (***ground vibration test***)on the ground prior to actual flight of the vehicle.

VIBRATION TESTING OF STRUCTURES

An introduction to *Experimental Modal Analysis is provided.*

- A very important structural dynamics test procedure that is used extensively in the automotive and aerospace industries and is also used to test buildings, bridges, and other civil structures.



SCOPE OF THE COURSE

MULTI-DEGREE-OF-FREEDOM SYSTEMS

- Mathematical Models of MDOF Systems
- Vibration Properties of MDOF Systems: Modes, Frequencies, and Damping
- Dynamic Response of MDOF Systems: Mode-Superposition Method

DISTRIBUTED-PARAMETER SYSTEMS

- Mathematical Models of Continuous Systems
- Free Vibration of Continuous Systems
- Analysis of Dynamic Response
- Component-Mode Synthesis

Advanced Topics in Structural Dynamics

- Introduction to Experimental Modal Analysis
- Stochastic Response of Linear MDOF Systems

Text and Evaluation Scheme

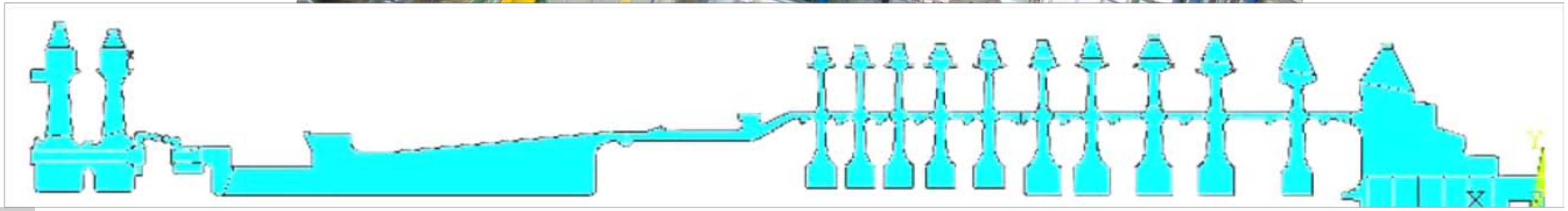
Textbook:

> **Fundamentals of Structural Dynamics**
> **Roy R. Craig, Jr., and Andrew J. Kurdila**
> 2nd Edition, **John Wiley & Sons, 2006.**

Evaluation Scheme:

> Three quizzes 30%
> Course project 30%
> Final exam 40%

COURSE PROJECT: Rotor dynamic analysis of gas turbine rotor



Structural Dynamics

Leacture Two: Mathematical Models of MDOF Systems (Chapter 8)

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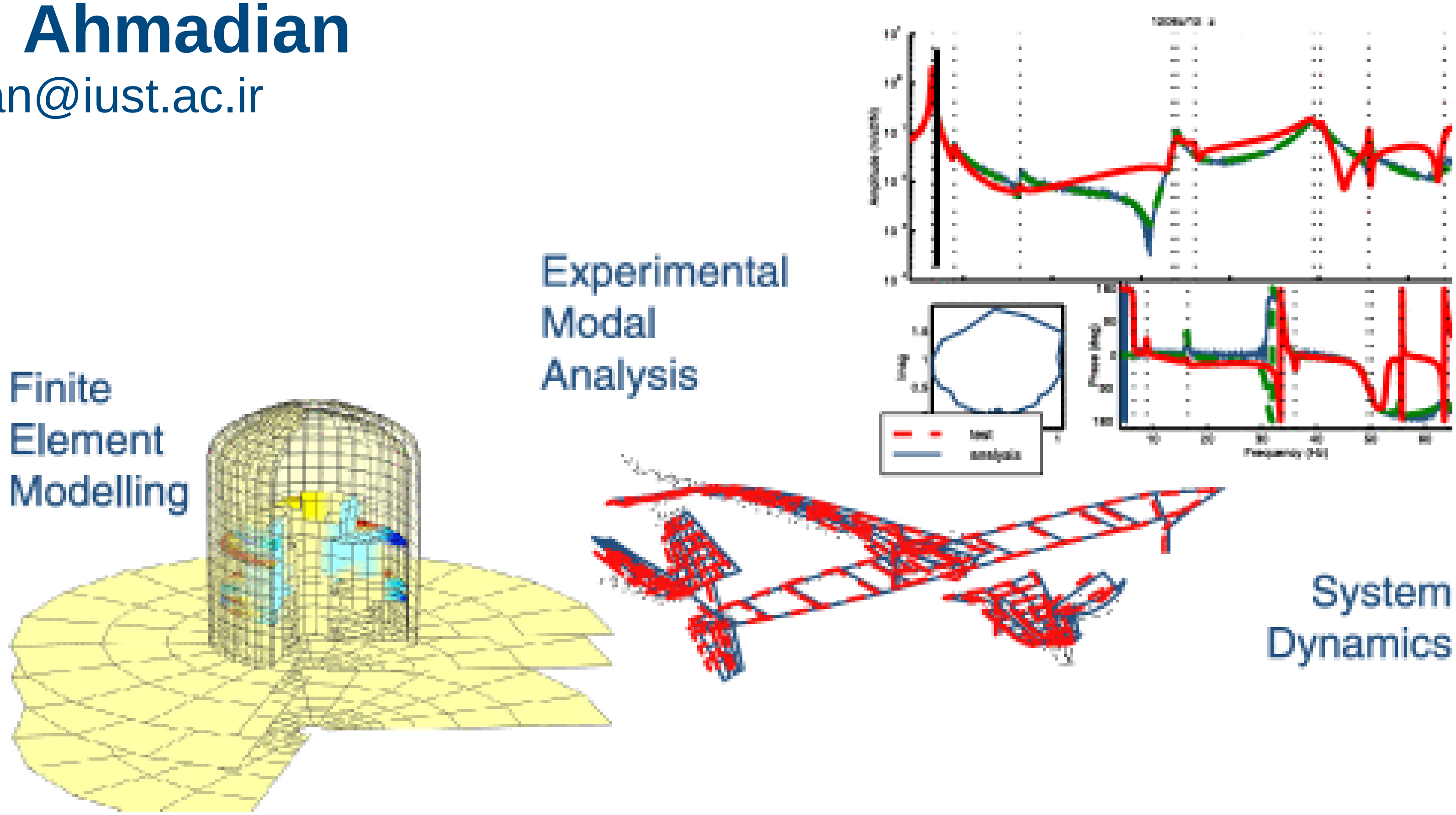

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Mathematical Models of MDOF Systems

The general form of the equations of motion of a linear *N* -DOF model of a structure is:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{p}(t)$$

where ***M*** is the *mass matrix,* ***C*** is the *viscous damping matrix,* and ***K*** is the *stiffness matrix.*

These coefficient matrices are all **N x N matrices**.

The *displacement vector* ***u(t),*** either physical or generalized displacements, and the corresponding *load vector* ***p(t)*** are **N x 1 vectors**.

APPLICATION OF NEWTON'S LAWS TO LUMPED-PARAMETER MODELS

Use Newton's Laws to derive the equations of motion of the system shown in Fig. 1. Express the equations of motion in terms of the displacements of the masses relative to the base.

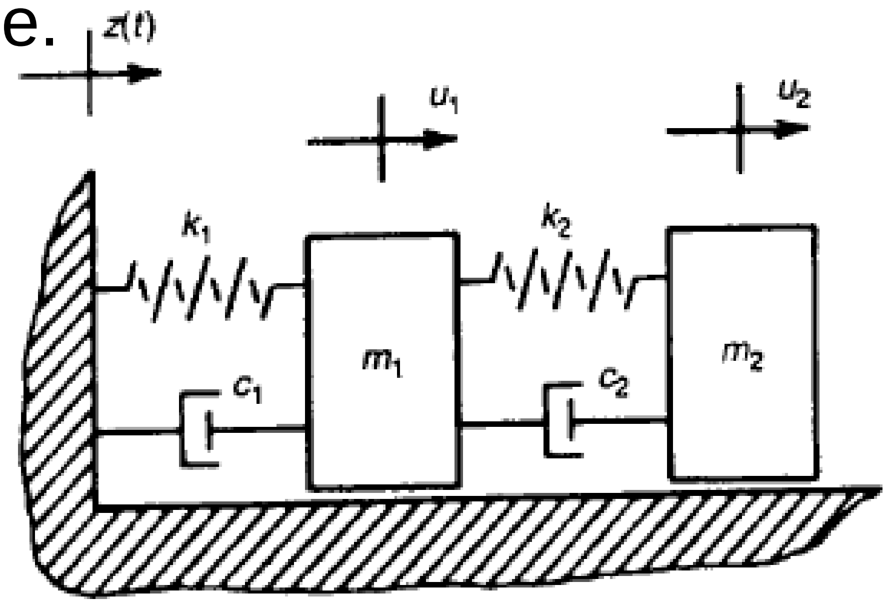


Figure 1 A 2-DOF spring–mass–dashpot system.

APPLICATION OF NEWTON'S LAWS TO LUMPED-PARAMETER MODELS

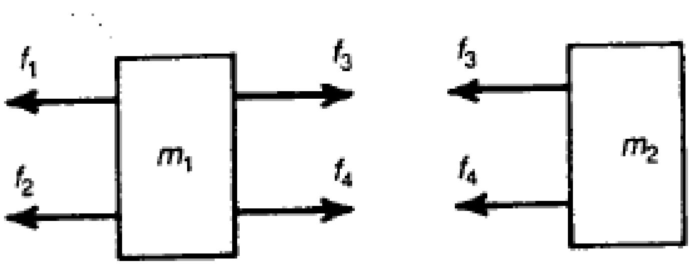


Figure 2 Free-body diagrams.

$$f_1 = k_1(u_1 - z) = k_1 w_1$$

$$f_2 = c_1(\dot{u}_1 - \dot{z}) = c_1 \dot{w}_1$$

$$f_3 = k_2(u_2 - u_1) = k_2(w_2 - w_1)$$

$$f_4 = c_2(\dot{u}_2 - \dot{u}_1) = c_2(\dot{w}_2 - \dot{w}_1)$$

$$\overset{+}{\rightarrow} \sum F_1 = m_1 \ddot{u}_1 = m_1(\ddot{z} + \ddot{w}_1) = -f_1 - f_2 + f_3 + f_4$$

$$\overset{+}{\rightarrow} \sum F_2 = m_2 \ddot{u}_2 = m_1(\ddot{z} + \ddot{w}_2) = -f_3 - f_4$$

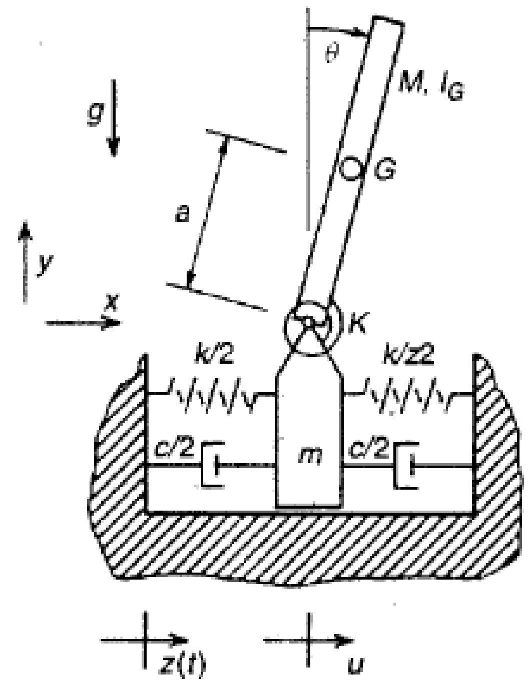
$$m_1 \ddot{w}_1 + c_1 \dot{w}_1 + k_1 w_1 - c_2(\dot{w}_2 - \dot{w}_1) - k_2(w_2 - w_1) = -m_1 \ddot{z}$$

$$m_2 \ddot{w}_2 + c_2(\dot{w}_2 - \dot{w}_1) + k_2(w_2 - w_1) = -m_2 \ddot{z}$$

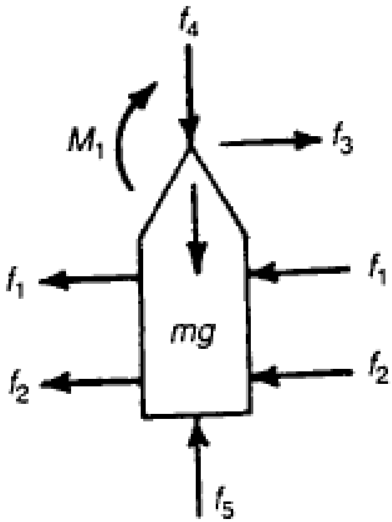
$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{w}_1 \\ \ddot{w}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \begin{Bmatrix} -m_1 \ddot{z} \\ -m_2 \ddot{z} \end{Bmatrix}$$

APPLICATION OF NEWTON'S LAWS TO LUMPED-PARAMETER MODELS

The motion of a mobile launcher subjected to base excitation is to be studied by using the lumped-parameter model shown. Use Newton's Laws to derive the equations of motion of this system.



APPLICATION OF NEWTON'S LAWS TO LUMPED-PARAMETER MODELS

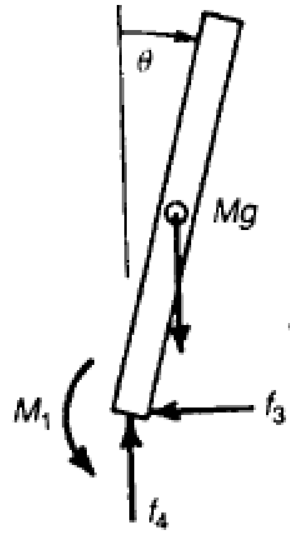


$$\overset{+}{\rightarrow} \sum F_x = m\ddot{u} = -2f_1 - 2f_2 + f_3$$

$$f_1 = \frac{k}{2}(u - z)$$

$$f_2 = \frac{c}{2}(\dot{u} - \dot{z})$$

$$M_1 = K\theta$$



$$\overset{+}{\rightarrow} \sum F_x = M\ddot{x}_G = -f_3$$

$$\uparrow + \sum F_y = M\ddot{y}_G = f_4 - Mg$$

$$\circlearrowleft + \sum M_G = I_G\ddot{\theta} = -M_1 + f_4 a \sin\theta + f_3 a \cos\theta$$

$$x_G = u + a\theta$$

$$y_G = a$$

$$\begin{bmatrix} M+m & Ma \\ Ma & I_G + Ma^2 \end{bmatrix} \begin{Bmatrix} \ddot{u} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & K - Mga \end{bmatrix} \begin{Bmatrix} u \\ \theta \end{Bmatrix} = \begin{Bmatrix} c\dot{z} + kz \\ 0 \end{Bmatrix}$$

INTRODUCTION TO ANALYTICAL DYNAMICS: HAMILTON'S PRINCIPLE AND LAGRANGE'S EQUATIONS

The study of dynamics may be subdivided into two main categories:

- ***Newtonian Mechanics*** and
- ***Analytical Mechanics*** /referred to as ***Variational Principles*** *in Mechanics* or ***Energy Methods*** *in Mechanics*

The Principle of ***Virtual Displacements***, ***Hamilton's Principle***, and **Lagrange's Equations** are analytical mechanics methods that are used to derive the equations of motion for various models of dynamical systems.

Hamilton's Principle

The *Extended Hamilton's Principle* may be stated as follows:

The motion of the given system from time t_1, to t_2 is such that:

$$\boxed{\int_{t_1}^{t_2} \delta(\mathcal{T} - \mathcal{V})\, dt + \int_{t_1}^{t_2} \delta \mathcal{W}_{nc}\, dt = 0}$$

$\mathcal{T}$ = total kinetic energy of the system

$\mathcal{V}$ = potential energy of the system, including the strain energy and the potential energy of conservative external forces

$\delta \mathcal{W}_{nc}$ = virtual work done by nonconservative forces, including damping forces and external forces not accounted for in $\mathcal{V}$

$\delta[\cdot]$ = symbol denoting the first variation, or virtual change, in the quantity in brackets

t_1, t_2 = times at which the configuration of the system is assumed to be known

Hamilton's Principle

For conservative systems $\delta \mathcal{W}_{nc} = 0$,

$$\int_{t_1}^{t_2} \delta \mathcal{L} \, dt = 0$$

$\mathcal{L} = \mathcal{T} - \mathcal{V}$ is called the ***Lagrangian function***. *The above e*quation is referred to as ***Hamilton's Principle***.

Lagrange's Equations

Generalized coordinates are defined as any set of N independent quantities that are sufficient to completely specify the position of every point within an N–DOF system.

$$\mathcal{T} = \mathcal{T}(q_1, q_2, \ldots, q_N, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_N, t)$$

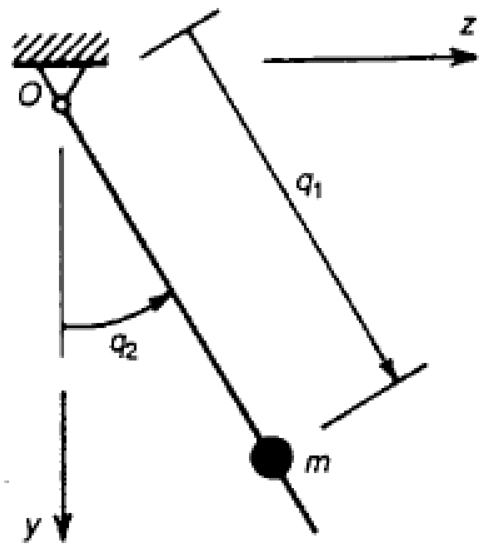
$$\mathcal{V} = \mathcal{V}(q_1, q_2, \ldots, q_N, t)$$

$$\delta \mathcal{W}_{\mathrm{nc}} = p_1 \delta q_1 + p_2 \delta q_2 + \cdots + p_N \delta q_N$$

where P_1 P_2, ... P_N are called the *generalized forces.*

Lagrange's Equations

A particle of mass ***m*** slides along a weightless rigid rod as shown. Write an expression for the kinetic energy of the particle in terms of the generalized coordinates q_1 and q_2 and their time derivatives.



$$T = \frac{1}{2} m (\dot{y}^2 + \dot{z}^2)$$

$$y = q_1 \cos q_2$$

$$z = q_1 \sin q_2$$

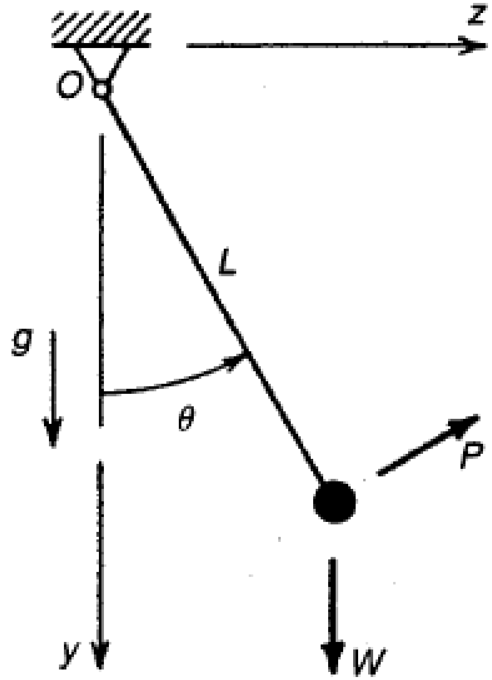
$$\dot{y} = \dot{q}_1 \cos q_2 - \dot{q}_2 q_1 \sin q_2$$

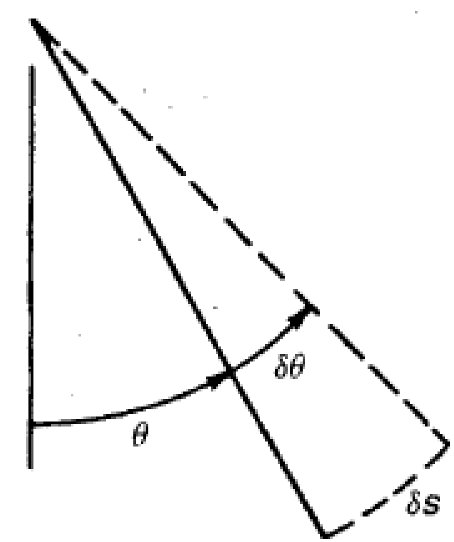
$$\dot{z} = \dot{q}_1 \sin q_2 + \dot{q}_2 q_1 \cos q_2$$

$$T = \frac{1}{2} m [\dot{q}_1^2 + (q_1 \dot{q}_2)^2]$$

Lagrange's Equations

A force *P* acts tangent to the path of a particle of weight *W*, which is attached to a rigid bar of length *L*. Obtain expressions for the potential energy of weight Wand the virtual work done by force *P*. Also, determine an expression for the generalized force.





$$\mathcal{V} = -Wy = -WL\cos\theta$$

$$\delta\mathcal{W}_P = P\,\delta s = (PL)\,\delta\theta$$

$$p_\theta = PL$$

Lagrange's Equations

$$\boxed{\int_{t_1}^{t_2} \delta(\mathcal{T}-\mathcal{V})\,dt + \int_{t_1}^{t_2} \delta\mathcal{W}_{nc}\,dt = 0}$$

$$\int_{t_1}^{t_2} \left(\frac{\partial \mathcal{T}}{\partial q_1}\delta q_1 + \frac{\partial \mathcal{T}}{\partial q_2}\delta q_2 + \cdots + \frac{\partial \mathcal{T}}{\partial q_N}\delta q_N + \frac{\partial \mathcal{T}}{\partial \dot{q}_1}\delta \dot{q}_1 + \frac{\partial \mathcal{T}}{\partial \dot{q}_2}\delta \dot{q}_2 \right.$$

$$+ \cdots + \frac{\partial \mathcal{T}}{\partial \dot{q}_N}\delta \dot{q}_N - \frac{\partial \mathcal{V}}{\partial q_1}\delta q_1 - \frac{\partial \mathcal{V}}{\partial q_2}\delta q_2 - \cdots - \frac{\partial \mathcal{V}}{\partial q_N}\delta q_N$$

$$\left. + p_1\delta q_1 + p_2\delta q_2 + \cdots + p_N\delta q_N \right) dt = 0$$

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{T}}{\partial \dot{q}_i}\delta \dot{q}_i\,dt = \left[\frac{\partial \mathcal{T}}{\partial \dot{q}_i}\delta q_i\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{q}_i}\delta q_i\,dt$$

$$\delta q_i(t_1) = \delta q_i(t_2) = 0$$

$$\int_{t_1}^{t_2} \left\{ \sum_{i=1}^{N} \left[-\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{q}_i} + \frac{\partial \mathcal{T}}{\partial q_i} - \frac{\partial \mathcal{V}}{\partial q_i} + p_i \right] \delta q_i \right\} dt = 0$$

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{q}_i} - \frac{\partial \mathcal{T}}{\partial q_i} + \frac{\partial \mathcal{V}}{\partial q_i} = p_i(t), \quad i = 1, 2, \ldots, N$$

APPLICATION OF LAGRANGE'S EQUATIONS TO LUMPED-PARAMETER MODELS

$$\mathcal{T} = 2\left(\tfrac{1}{2} m\, \dot{y}_m^2\right) + \tfrac{1}{2} M \dot{u}^2$$

$$y_m = u + L\theta$$

$$\mathcal{T} = m(\dot{u} + L\dot{\theta})^2 + \tfrac{1}{2} M \dot{u}^2$$

$$\mathcal{V} = 2\left(\tfrac{1}{2} k\, \theta^2\right)$$

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{q}_i} - \frac{\partial \mathcal{T}}{\partial q_i} + \frac{\partial \mathcal{V}}{\partial q_i} = p_i,$$

$$q_1 \to u, \quad q_2 \to \theta$$

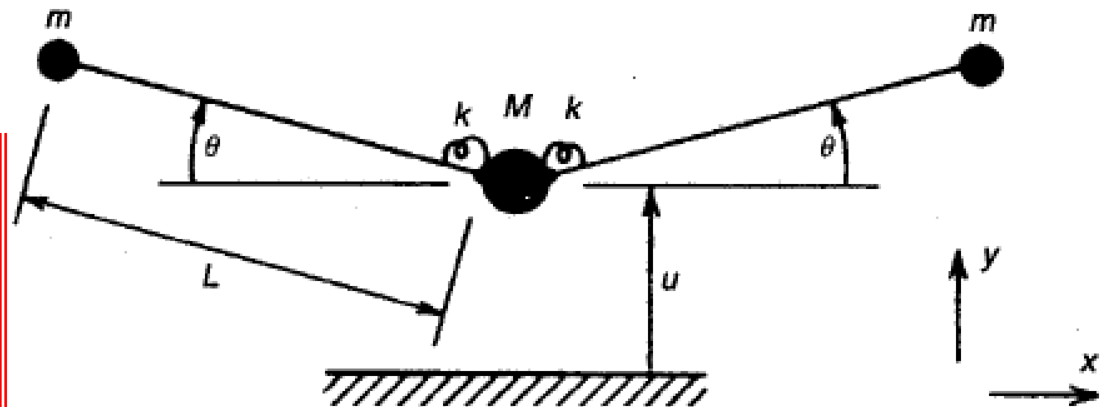


Figure 1 A 2-DOF model of an airplane.

$$\frac{\partial \mathcal{T}}{\partial \dot{u}} = 2m(\dot{u} + L\dot{\theta}) + M\dot{u} \qquad \frac{\partial \mathcal{T}}{\partial \dot{\theta}} = 2mL(\dot{u} + L\dot{\theta})$$

$$\frac{\partial \mathcal{T}}{\partial u} = \frac{\partial \mathcal{T}}{\partial \theta} = 0 \qquad \frac{\partial \mathcal{V}}{\partial u} = 0, \qquad \frac{\partial \mathcal{V}}{\partial \theta} = 2k\theta$$

$$p_u = p_\theta = 0$$

$$\begin{bmatrix} M + 2m & 2mL \\ 2mL & 2mL^2 \end{bmatrix} \begin{Bmatrix} \ddot{u} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2k \end{bmatrix} \begin{Bmatrix} u \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

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Leacture Three: Mathematical Models of MDOF Systems (Chapter 8)

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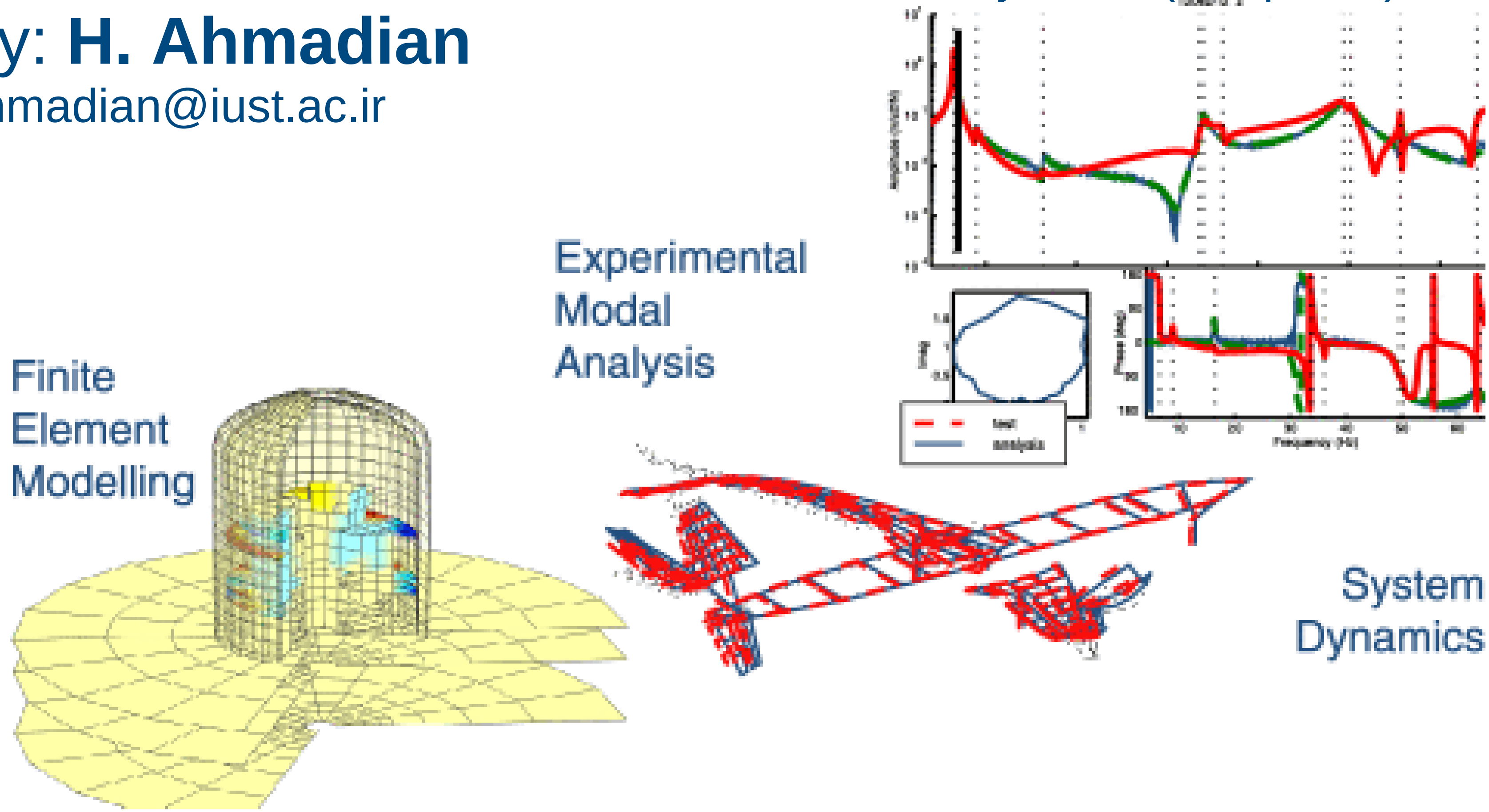

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APPLICATION OF LAGRANGE'S EQUATIONS TO CONTINUOUS MODELS

- ASSUMED-MODES METHOD
  - Selection of Shape Functions
  - Cantilever Beam Example
  - Axial vibration of a linearly elastic bar
- Procedure for the Assumed-Modes Method
  - A 2-DOF model for axial vibration of a uniform cantilever bar
  - Assumed-Modes Method: Bending of Bernoulli-Euler Beams
  - A missile on launch pad

APPLICATION OF LAGRANGE'S EQUATIONS TO CONTINUOUS MODELS: ASSUMED-MODES METHOD

To generate an N DOF model of a continuous system, the continuous displacement ***u(x,t)*** is approximated by the finite sum:

$$\boxed{u(x,t) = \sum_{i=1}^{N} \psi_i(x) q_i(t)}$$

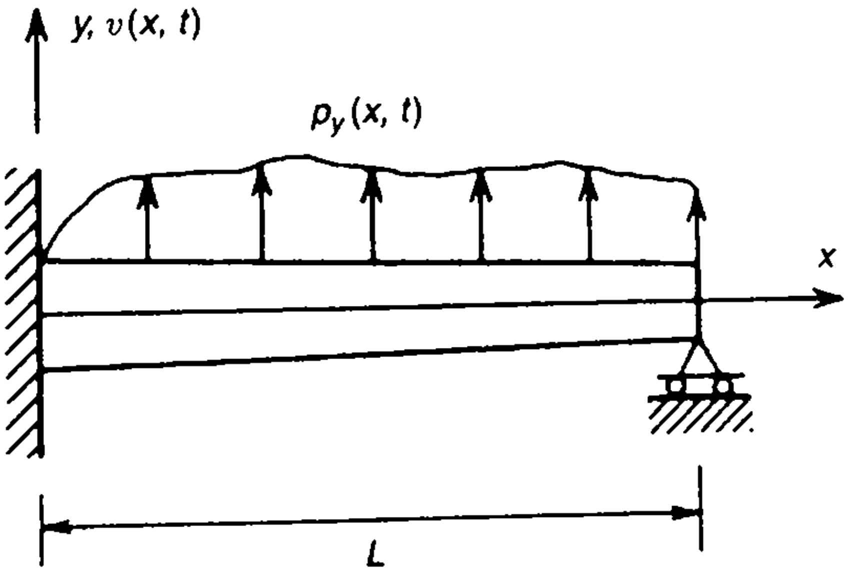
Selection of Shape Functions

The ***shape functions*** *must:*

- Form a linearly ***independent*** set.
- Possess derivatives up to the order appearing in the strain energy ***V****.*
- Satisfy all *prescribed* ***boundary conditions****,* that is, all displacement-type boundary conditions.

Functions that satisfy these three conditions are called ***admissible functions.***

Cantilever Beam Example



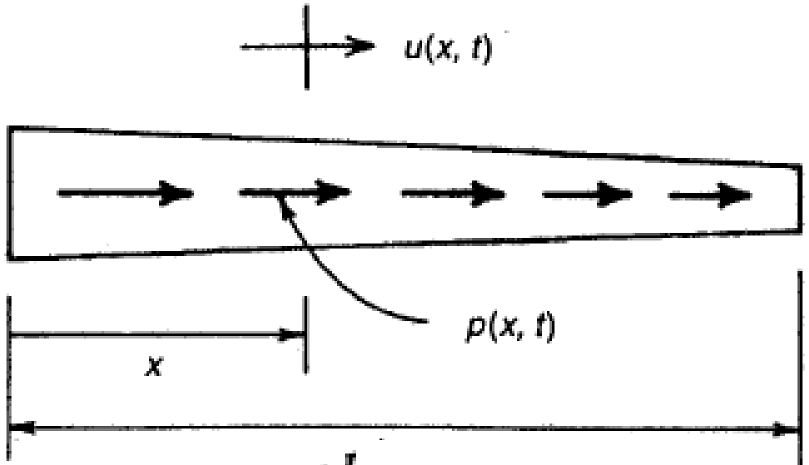
$$v(x,t) = \sum_{i=1}^{N} \psi_i(x) q_i(t)$$

$$v(0,t) = v'(0,t) = v(L,t) = 0$$

$$\longrightarrow \quad \psi_i(0) = \psi_i'(0) = \psi_i(L) = 0$$

Since the strain energy for a Bernoulli-Euler beam contains the second derivative of the transverse displacement, assumed modes must be continuous functions of *x,* and its first derivative.

Axial vibration of a linearly elastic bar



$$\boxed{u(x,t) = \sum_{i=1}^{N} \psi_i(x) u_i(t)}$$

$$\mathcal{V} = \frac{1}{2}\int_0^L EA(u')^2\,dx$$

$$\mathcal{T} = \frac{1}{2}\int_0^L \rho A(\dot{u})^2\,dx$$

$$\mathcal{V} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} k_{ij} u_i u_j$$

$$\mathcal{T} = \frac{1}{2}\sum_{i=1}^{N}\sum_{i=1}^{N} m_{ij} \dot{u}_i \dot{u}_j$$

$$\boxed{k_{ij} = \int_0^L EA\psi_i'\psi_j'\,dx}$$

$$\boxed{m_{ij} = \int_0^L \rho A\psi_i\psi_j\,dx}$$

$$\mathcal{V} = \frac{1}{2}\mathbf{u}^{\mathsf{T}}\mathbf{K}\mathbf{u}$$

$$\mathcal{T} = \frac{1}{2}\dot{\mathbf{u}}^{\mathsf{T}}\mathbf{M}\dot{\mathbf{u}}$$

Axial vibration of a linearly elastic bar

$$\delta \mathcal{W} = \int_0^L p(x,t)\,\delta u(x,t)\,dx = \sum_{i=1}^{N} p_i(t)\,\delta u_i \qquad \delta u(x,t) = \sum_{i=1}^{N} \psi_i(x)\,\delta u_i$$

$$\boxed{p_i(t) = \int_0^L p(x,t)\psi_i(x)\,dx}$$

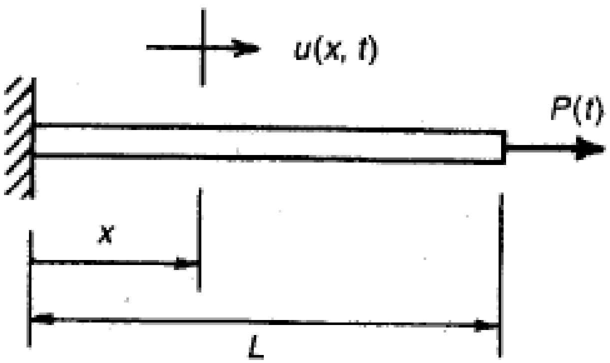
$$\sum_{j=1}^{N} m_{ij}\ddot{u}_j + \sum_{j=1}^{N} k_{ij}u_j = p_i(t), \qquad i = 1, 2, \ldots, N$$

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{p}(t)$$

Procedure for the Assumed-Modes Method

1. Select a set of N admissible functions.
2. Compute the coefficients k_{ij} of the stiffness matrix.
3. Compute the coefficients m_{ij} of the mass matrix
4. Determine expressions for the generalized forces $P_i(t)$ corresponding to the applied force $p(x,t)$.
5. Form the equations of motion.

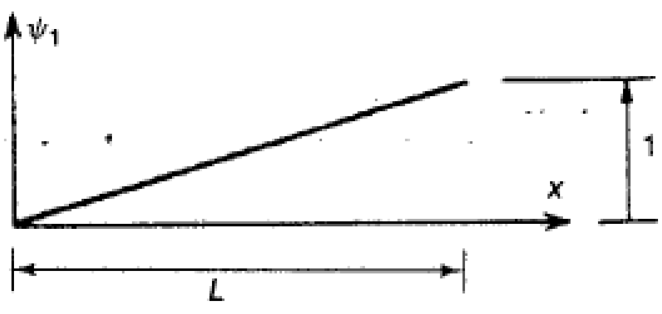
A 2-DOF model for axial vibration of a uniform cantilever bar

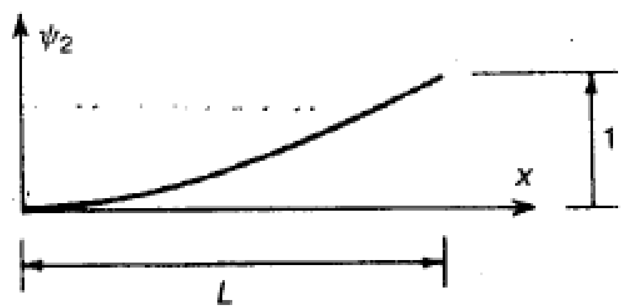


$$u(x,t) = \psi_1(x)\,u_1(t) + \psi_2(x)\,u_2(t)$$

$$u(0,t) = 0 \quad \Longrightarrow \quad \psi_1(0) = \psi_2(0) = 0$$

$$\psi_1(x) = \frac{x}{L}, \qquad \psi_2(x) = \left(\frac{x}{L}\right)^2$$





A 2-DOF model for axial vibration of a uniform cantilever bar

$$k_{11} = \int_0^L EA(\psi_1')^2\,dx = \frac{EA}{L}$$

$$k_{12} = k_{21} = \int_0^L EA\psi_1'\psi_2'\,dx = \frac{EA}{L}$$

$$k_{22} = \int_0^L EA(\psi_2')^2\,dx = \frac{4EA}{3L}$$

$$m_{11} = \int_0^L \rho A(\psi_1)^2\,dx = \frac{\rho AL}{3}$$

$$m_{12} = m_{21} = \int_0^L \rho A\psi_1\psi_2\,dx = \frac{\rho AL}{4}$$

$$m_{22} = \int_0^L \rho A(\psi_2)^2\,dx = \frac{\rho AL}{5}$$

$$\delta W = P(t)\,\delta u(L,t) = p_1\,\delta u_1 + p_2\,\delta u_2$$

$$\delta u(L,t) = \psi_1(L)\,\delta u_1 + \psi_2(L)\,\delta u_2$$

$$p_1(t) = P(t)\,\psi_1(L) = P(t),$$

$$p_2(t) = P(t)\,\psi_2(L) = P(t)$$

$$\rho AL\begin{bmatrix}\frac{1}{3} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{5}\end{bmatrix}\begin{Bmatrix}\ddot{u}_1\\ \ddot{u}_2\end{Bmatrix} + \frac{EA}{L}\begin{bmatrix}1 & 1\\ 1 & \frac{4}{3}\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}P(t)\\ P(t)\end{Bmatrix}$$

Assumed-Modes Method: Bending of Bernoulli-Euler Beams

$$\mathcal{V} = \frac{1}{2} \int_0^L EI\,(v'')^2\,dx$$

$$\mathcal{T} = \frac{1}{2} \int_0^L \rho A(\dot{v})^2\,dx$$

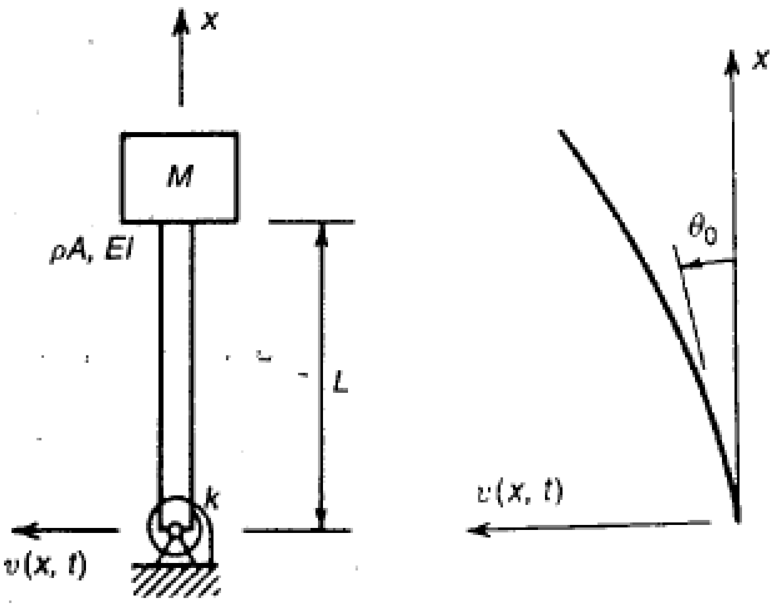
$$\boxed{v(x,t) = \sum_{i=1}^{N} \psi_i(x) v_i(t)}$$

$$\boxed{k_{ij} = \int_0^L EI\,\psi_i''\psi_j''\,dx}$$

$$\boxed{m_{ij} = \int_0^L \rho A \psi_i \psi_j\,dx}$$

$$\boxed{p_i(t) = \int_0^L p_y(x,t)\psi_i(x)\,dx}$$

A missile on launch pad



$$v(x,t) = \psi_1(x)v_1(t) + \psi_2(x)v_2(t)$$

$$v(0,t) = 0 \longrightarrow \psi_1(0) = \psi_2(0) = 0$$

$$\psi_1(x) = \frac{x}{L}, \qquad \psi_2(x) = \left(\frac{x}{L}\right)^2$$

A missile on launch pad

$$\mathcal{V} = \tfrac{1}{2}\int_0^L EI\,(v'')^2\,dx + \tfrac{1}{2}k\,\theta_0^2$$

$$\theta_0(t) \approx v'(0,t) = \psi_1'(0)v_1(t) + \psi_2'(0)v_2(t)$$

$$k_{ij} = \int_0^L EI\,\psi_i''\psi_j''\,dx + k\,\psi_i'(0)\psi_j'(0)$$

$$\psi_1' = \frac{1}{L}, \qquad \psi_1'' = 0$$

$$\psi_2' = \frac{2}{L}\frac{x}{L}, \qquad \psi_2'' = \frac{2}{L^2}$$

$$k_{11} = 0 + k\,\frac{1}{L}\frac{1}{L} = \frac{k}{L^2}$$

$$k_{12} = k_{21} = 0 + 0 = 0$$

$$k_{22} = \frac{4EI}{L^3} + 0 = \frac{4EI}{L^3}$$

$$m_{ij} = \int_0^L \rho A\psi_i\psi_j\,dx + M\psi_i(L)\psi_j(L)$$

$$m_{11} = \rho A\int_0^L \left(\frac{x}{L}\right)^2 dx + M(1)(1) = \frac{\rho AL}{3} + M$$

$$m_{12} = m_{21} = \frac{\rho AL}{4} + M$$

$$m_{22} = \frac{\rho AL}{5} + M$$

$$\boxed{\begin{bmatrix} \dfrac{\rho AL}{3} + M & \dfrac{\rho AL}{4} + M \\ \dfrac{\rho AL}{4} + M & \dfrac{\rho AL}{5} + M \end{bmatrix} \begin{Bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \end{Bmatrix} + \begin{bmatrix} \dfrac{k}{L^2} & 0 \\ 0 & \dfrac{4EI}{L^3} \end{bmatrix} \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

Structural Dynamics

Lecture Four: Mathematical Models of MDOF Systems (Chapter 8)

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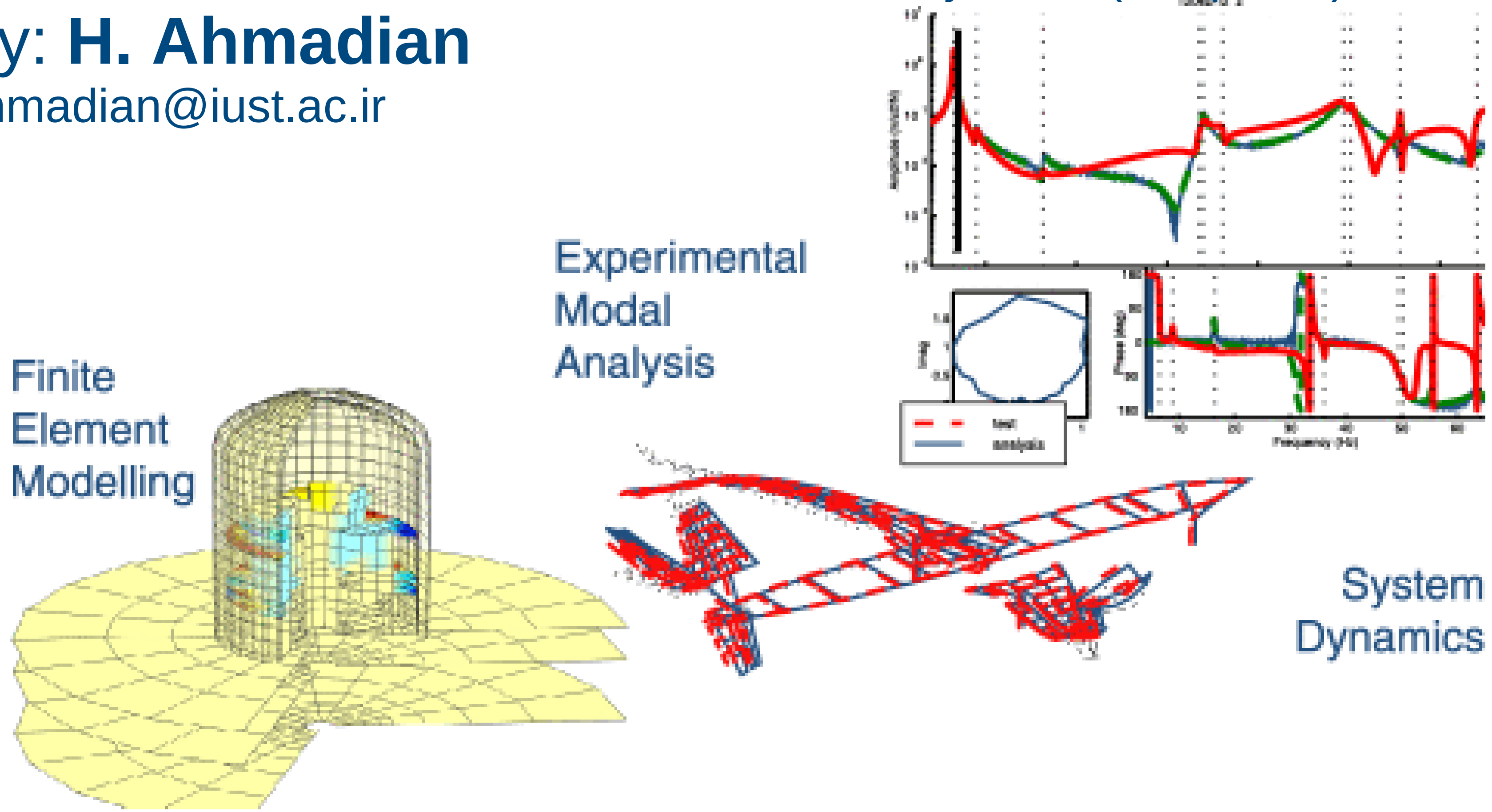

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SUMMERY

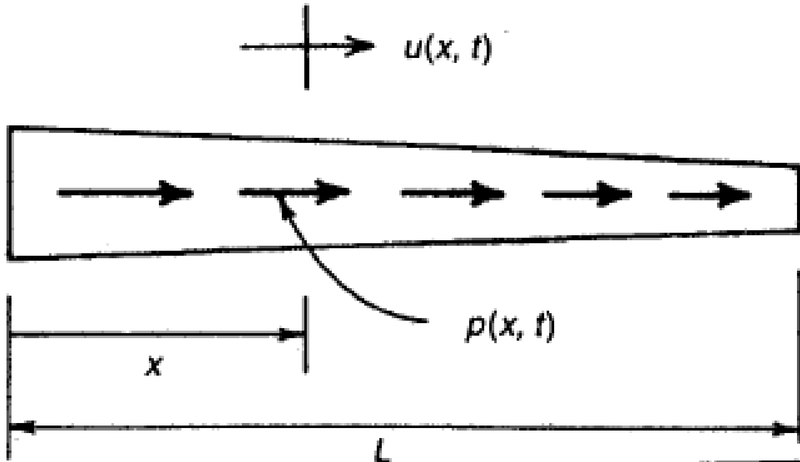
Application of Lagrange's Equations to continuous models:

> ASSUMED-MODES METHOD

To generate an *N* DOF model of a continuous system, the continuous displacement ***u(x,t)*** is approximated by the finite sum:

$$\boxed{u(x,t) = \sum_{i=1}^{N} \psi_i(x) q_i(t)}$$

Axial vibration of a linearly elastic bar



$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{q}_i} - \frac{\partial \mathcal{T}}{\partial q_i} + \frac{\partial \mathcal{V}}{\partial q_i} = p_i(t),$$

$$i = 1, 2, \ldots, N$$

$$\boxed{u(x,t) = \sum_{i=1}^{N} \psi_i(x) u_i(t)}$$

$$\mathcal{V} = \tfrac{1}{2}\int_0^L EA(u')^2\,dx$$

$$\mathcal{V} = \tfrac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} k_{ij} u_i u_j$$

$$\boxed{k_{ij} = \int_0^L EA\psi_i'\psi_j'\,dx}$$

$$\mathcal{V} = \tfrac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u}$$

$$\mathcal{T} = \tfrac{1}{2}\int_0^L \rho A(\dot{u})^2\,dx$$

$$\mathcal{T} = \tfrac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} m_{ij}\dot{u}_i\dot{u}_j$$

$$\boxed{m_{ij} = \int_0^L \rho A\psi_i\psi_j\,dx}$$

$$\mathcal{T} = \tfrac{1}{2}\dot{\mathbf{u}}^T\mathbf{M}\dot{\mathbf{u}}$$

Axial vibration of a linearly elastic bar

$$\delta \mathcal{W} = \int_0^L p(x,t)\,\delta u(x,t)\,dx = \sum_{i=1}^{N} p_i(t)\,\delta u_i \qquad \delta u(x,t) = \sum_{i=1}^{N} \psi_i(x)\,\delta u_i$$

$$\boxed{p_i(t) = \int_0^L p(x,t)\psi_i(x)\,dx}$$

$$\sum_{j=1}^{N} m_{ij}\ddot{u}_j + \sum_{j=1}^{N} k_{ij}u_j = p_i(t), \qquad i = 1, 2, \ldots, N$$

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{p}(t)$$

Assumed-Modes Method: Bending of Bernoulli-Euler Beams

$$\mathcal{V} = \frac{1}{2}\int_0^L EI\,(v'')^2\,dx$$

$$\mathcal{T} = \frac{1}{2}\int_0^L \rho A(\dot{v})^2\,dx$$

$$\boxed{v(x,t) = \sum_{i=1}^{N} \psi_i(x)\,v_i(t)}$$

$$\boxed{k_{ij} = \int_0^L EI\,\psi_i''\psi_j''\,dx}$$

$$\boxed{m_{ij} = \int_0^L \rho A\,\psi_i\,\psi_j\,dx}$$

$$\boxed{p_i(t) = \int_0^L p_y(x,t)\,\psi_i(x)\,dx}$$

Other Effects: Distributed Viscous Damping

$$p(x,t) = -\xi(x)\dot{v}(x,t)$$

$$p_i = \int_0^L p(x,t)\psi_i(x)\,dx$$

$$= \int_0^L \left[-\xi(x)\sum_{j=1}^{N}\psi_j(x)\dot{v}_j(t)\right]\psi_i(x)\,dx$$

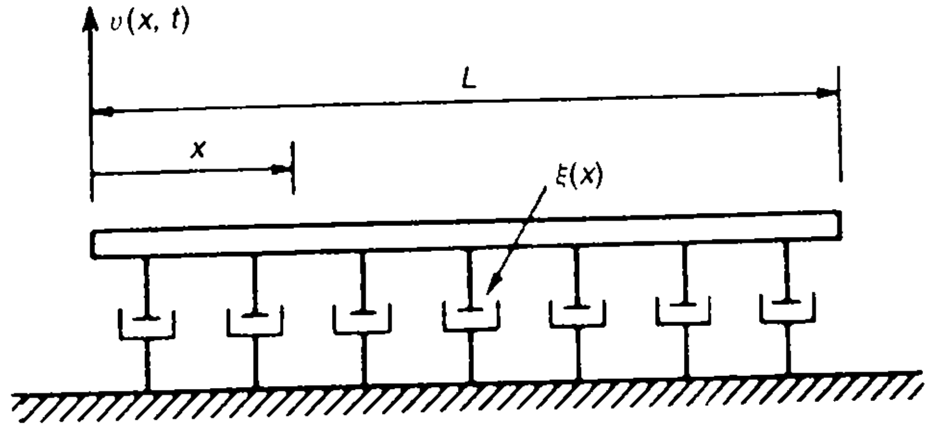


Figure 1 Distributed viscous damping acting on the beam.

$$p_i = -\sum_{j=1}^{N}\dot{v}_j(t)\left[\int_0^L \xi(x)\psi_i(x)\psi_j(x)\,dx\right]$$

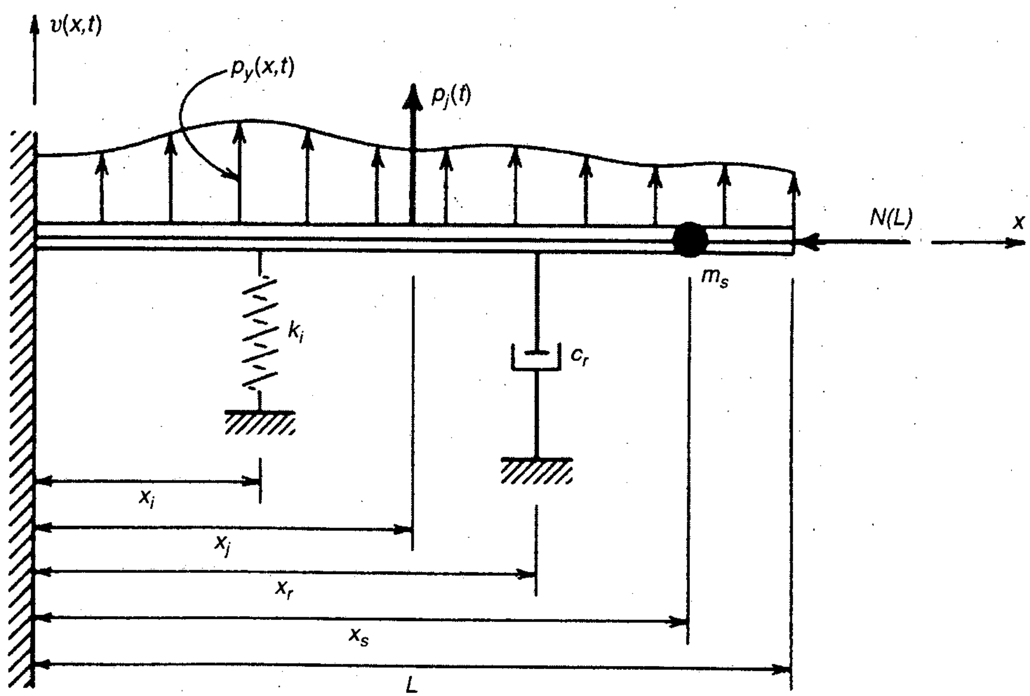
$$p_i = -\sum_{j=1}^{N} c_{ij}\dot{v}_j(t)$$

$$\boxed{\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{p}(t)}$$

$$\boxed{c_{ij} = \int_0^L \xi(x)\psi_i(x)\psi_j(x)\,dx}$$

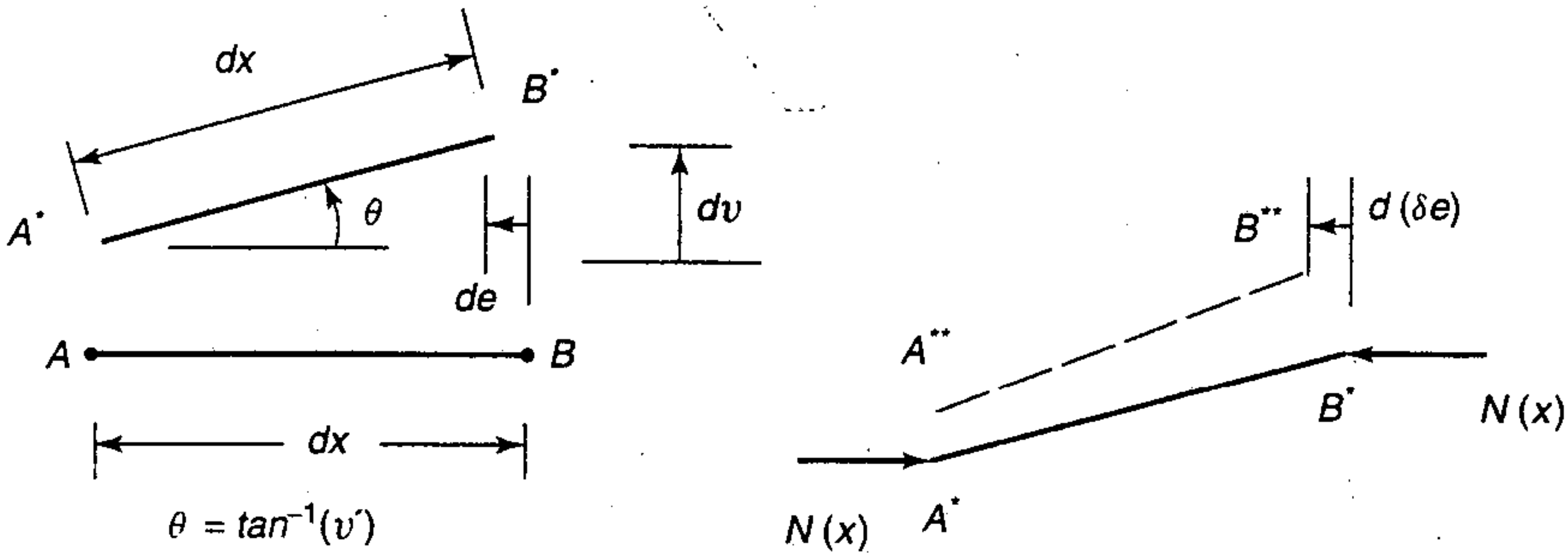
Other Effects: Geometric Stiffness

In a situation where a member is subjected to axial loading and also undergoes transverse deflection, the axial load may have a significant effect on the bending stiffness of the member.



Other Effects: Geometric Stiffness

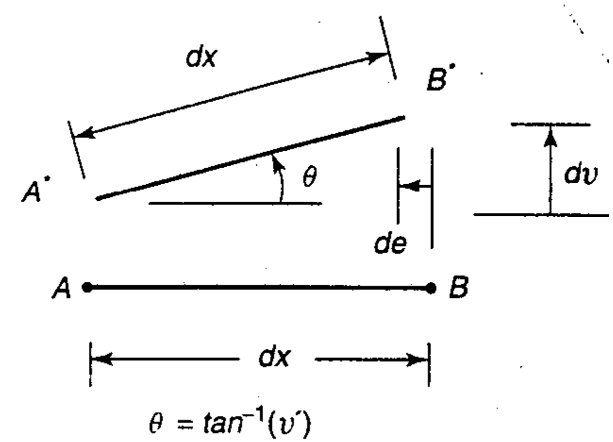
Using the virtual work done by the compressive axial force, *N(x),* the *generalized geometric stiffness coefficient* k_G is obtained:



$$d(\delta \mathcal{W}_N) = N(x)\, d(\delta e)$$

Other Effects: Geometric Stiffness

Assume that the element *AB* of length *dx* remains *dx* in length but rotates to the position *A*B** due to the transverse deflection:



$$\overline{AB} = \overline{A^*B^*}\cos(v') + de \qquad \cos v' \approx 1 - \tfrac{1}{2}(v')^2$$

$$dx = dx[1 - \tfrac{1}{2}(v')^2] + de$$

$$de = \tfrac{1}{2}(v')^2\,dx \longrightarrow d(\delta e) = \delta(de) = v'\,\delta v'\,dx$$

$$\delta W_N = \int_0^L N(x) v'\,\delta v'\,dx$$

Other Effects: Geometric Stiffness

$$\boxed{k_{Gij} = \int_0^L N(x)\,\psi_i'(x)\,\psi_j'(x)\,dx}$$

CONSTRAINED COORDINATES AND LAGRANGE MULTIPLIERS

Occasionally, it is desirable to employ a set of coordinates that are not independent.

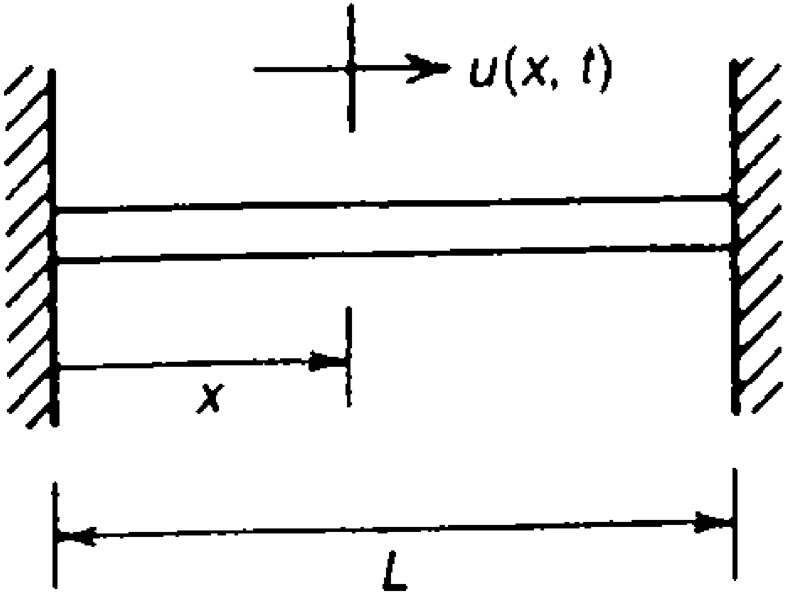
Let these be denoted by g_1, g_2,... , g_M, where $M > N$ *and a*ssociated C = *M - N constraint equations.*

Let these constraint equations be written in the form

$$\boxed{f_j(g_1, g_2, \ldots, g_M) = 0, \qquad j = 1, 2, \ldots, C}$$

$$\delta f_j = \frac{\partial f_j}{\partial g_1}\delta g_1 + \frac{\partial f_j}{\partial g_2}\delta g_2 + \cdots + \frac{\partial f_j}{\partial g_M}\delta g_M = 0$$

CONSTRAINED COORDINATES



$$u(x,t) = \frac{x}{L} g_1 + \left(\frac{x}{L}\right)^2 g_2$$

$$u(0,t) = u(L,t) = 0$$

$$f(g_1, g_2) \equiv u(L,t) = g_1 + g_2 = 0$$

CONSTRAINED COORDINATES

$$\int_{t_1}^{t_2} \left\{ \sum_{i=1}^{M} \left[-\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{g}_i} + \frac{\partial \mathcal{T}}{\partial g_i} - \frac{\partial \mathcal{V}}{\partial g_i} + p_i \right] \delta g_i \right\} dt = 0$$

Since the δg's are not independent, we cannot just set the expression in brackets to zero.

$$\sum_{i=1}^{M} \frac{\partial f_j}{\partial g_i} \delta g_i = 0, \qquad j = 1, 2, \ldots, C \qquad \sum_{j=1}^{C} \lambda_j \sum_{i=1}^{M} \frac{\partial f_j}{\partial g_i} \delta g_i = 0$$

$$\int_{t_1}^{t_2} \left\{ \sum_{i=1}^{M} \left[-\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{g}_i} + \frac{\partial \mathcal{T}}{\partial g_i} - \frac{\partial \mathcal{V}}{\partial g_i} + p_i + \sum_{j=1}^{C} \lambda_j \frac{\partial f_j}{\partial g_i} \right] \delta g_i \right\} dt = 0$$

CONSTRAINED COORDINATES

$$\int_{t_1}^{t_2} \left\{ \sum_{i=1}^{M} \left[-\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{g}_i} + \frac{\partial \mathcal{T}}{\partial g_i} - \frac{\partial \mathcal{V}}{\partial g_i} + p_i + \sum_{j=1}^{C} \lambda_j \frac{\partial f_j}{\partial g_i} \right] \delta g_i \right\} dt = 0$$

We can choose the Lagrange multipliers so as to make the bracketed expressions for *i* = 1, 2,.... ,*C* equal to zero.

Thus, the bracketed expression must vanish for all δg's giving the following *Lagrange's Equations* modified for constrained coordinates:

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{g}_i} - \frac{\partial \mathcal{T}}{\partial g_i} + \frac{\partial \mathcal{V}}{\partial g_i} - \sum_{j=1}^{C} \lambda_j \frac{\partial f_j}{\partial g_i} = p_i. \qquad i = 1, 2, \ldots, M}$$

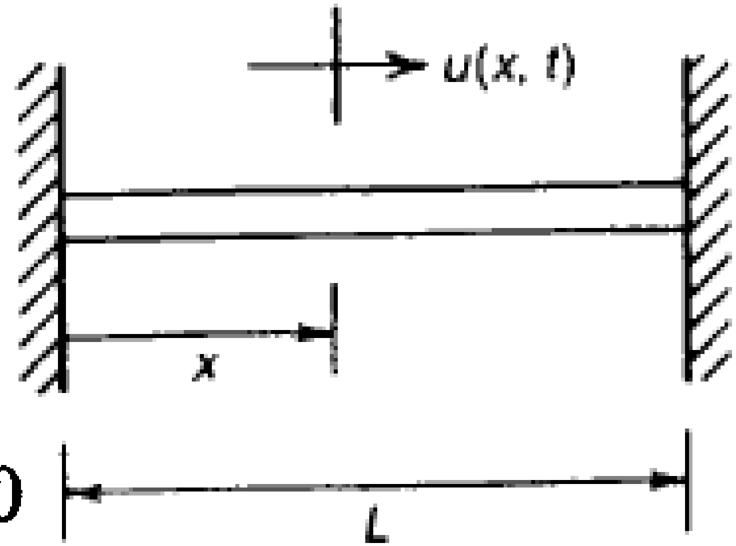
Alternative Lagrange's Equations

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{g}_i} - \frac{\partial \mathcal{T}}{\partial g_i} + \frac{\partial \mathcal{V}}{\partial g_i} - \sum_{j=1}^{C} \lambda_j \frac{\partial f_j}{\partial g_i} = p_i, \qquad i = 1, 2, \ldots, M}$$

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{g}_i} - \frac{\partial \mathcal{T}}{\partial g_i} + \frac{\partial \mathcal{V}^*}{\partial g_i} = p_i, \qquad i = 1, 2, \ldots, M$$

$$\mathcal{V}^* = \mathcal{V} - \sum_{j=1}^{C} \lambda_j f_j$$

Example:



$$u(x,t) = \frac{x}{L} g_1 + \left(\frac{x}{L}\right)^2 g_2$$

$$u(0,t) = u(L,t) = 0$$

$$f(g_1, g_2) \equiv u(L,t) = g_1 + g_2 = 0$$

$$\boxed{\frac{d}{dt}\frac{\partial T}{\partial \dot{g}_i} - \frac{\partial T}{\partial g_i} + \frac{\partial V}{\partial g_i} - \sum_{j=1}^{C} \lambda_j \frac{\partial f_j}{\partial g_i} = p_i, \qquad i = 1, 2, \ldots, M}$$

$$\rho A L \begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{Bmatrix} \ddot{g}_1 \\ \ddot{g}_2 \end{Bmatrix} + \frac{EA}{L} \begin{bmatrix} 1 & 1 \\ 1 & \frac{4}{3} \end{bmatrix} \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix} - \begin{Bmatrix} \lambda \\ \lambda \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Example:

$$f(g_1, g_2) \equiv u(L,t) = g_1 + g_2 = 0$$

$$\rho AL \begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{Bmatrix} \ddot{g}_1 \\ \ddot{g}_2 \end{Bmatrix} + \frac{EA}{L} \begin{bmatrix} 1 & 1 \\ 1 & \frac{4}{3} \end{bmatrix} \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix} - \begin{Bmatrix} \lambda \\ \lambda \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\frac{\rho AL}{10} \ddot{g}_1 + \frac{EA}{L} g_1 = 0$$

$$u(x,t) = \left[\frac{x}{L} - \left(\frac{x}{L} \right)^2 \right] g_1(t)$$

Structural Dynamics

Lecture Five: **Vibration of Undamped 2-DOF Systems** (Chapter 9)

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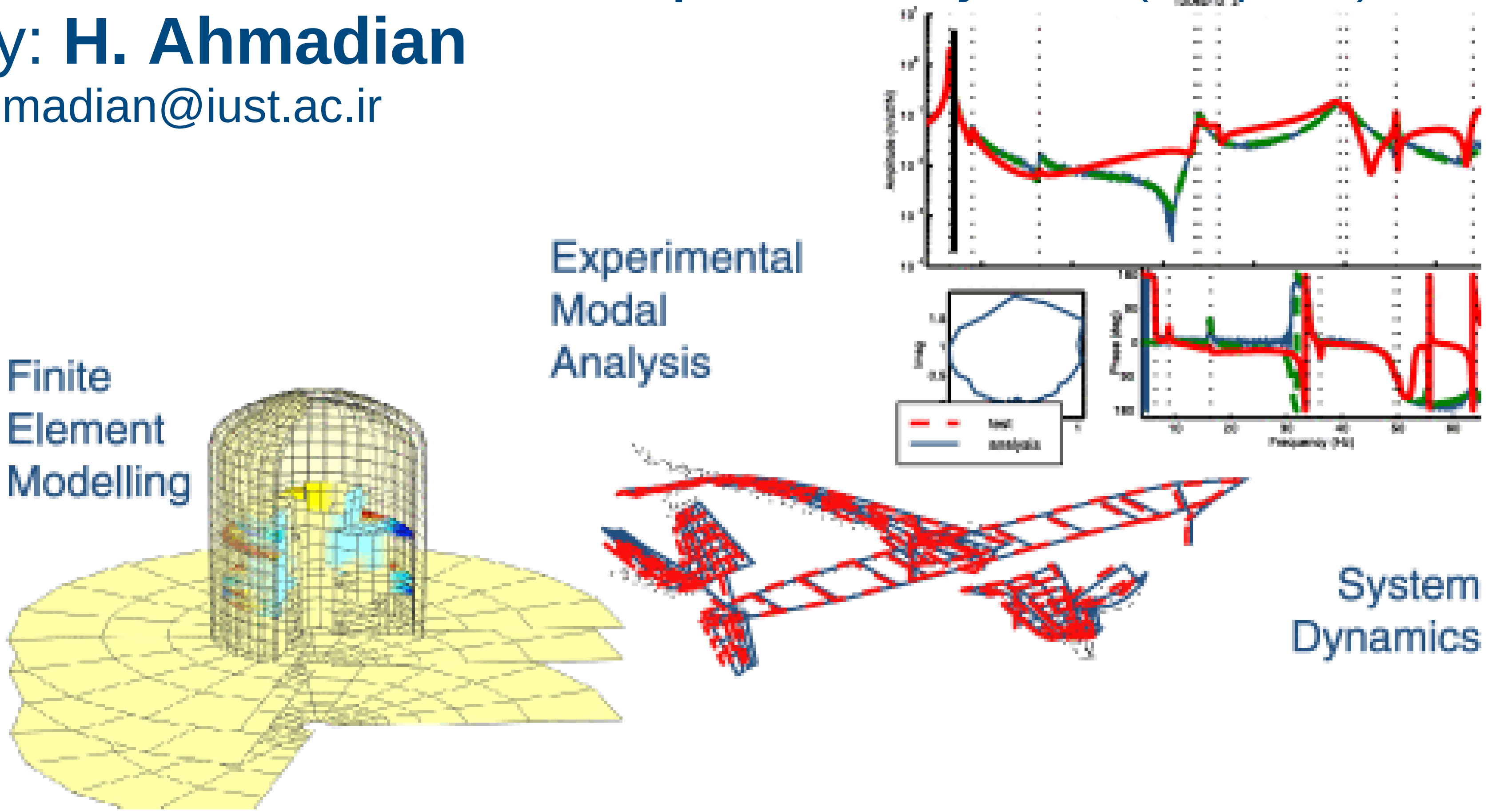

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Vibration of Undamped 2-DOFSystems

1. The natural frequencies and natural modes of a system.
2. Expressions for the response to initial conditions. The beat phenomenon.
3. The natural frequencies/natural modes of systems.
4. The modal matrix, the modal stiffness matrix, and the modal mass matrix.
5. Expressions for the steady-state frequency response in principal coordinates and in physical coordinates.

> An undamped vibration absorber.

FREE VIBRATION : NATURAL FREQUENCIES AND MODE SHAPES

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$u_1(t) = U_1 \cos(\omega t - \alpha)$$

$$u_2(t) = U_2 \cos(\omega t - \alpha)$$

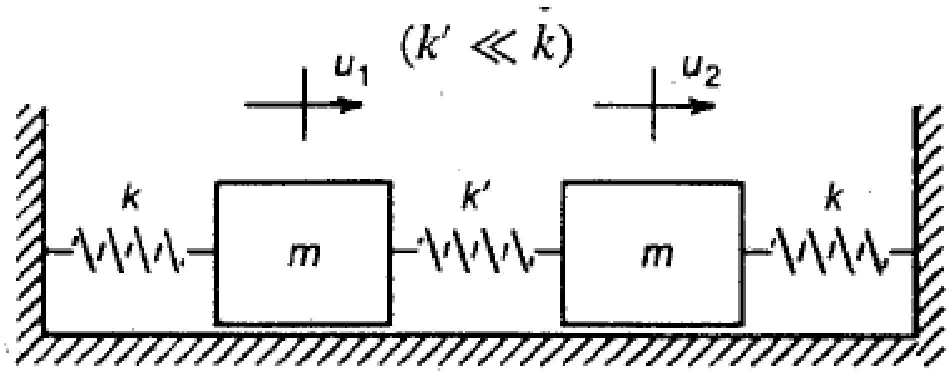
$$\boxed{\left[\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \right] \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

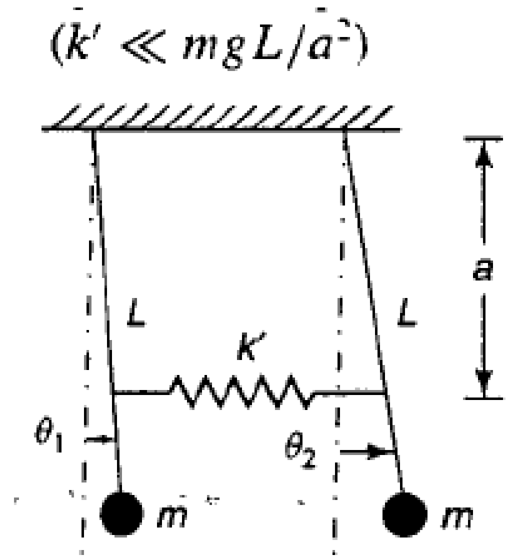
$$\boxed{\left| \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \omega_i^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \right| = 0}$$

$$f_1 = \frac{\omega_1}{2\pi}, \qquad f_2 = \frac{\omega_2}{2\pi} \qquad \boldsymbol{\phi}_r \equiv \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix}_r = \begin{Bmatrix} 1 \\ \beta_r \end{Bmatrix}, \qquad r = 1, 2$$

$$\mathbf{u}(t) \equiv \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = A_1 \boldsymbol{\phi}_1 \cos \omega_1 t + B_1 \boldsymbol{\phi}_1 \sin \omega_1 t + A_2 \boldsymbol{\phi}_2 \cos \omega_2 t + B_2 \boldsymbol{\phi}_2 \sin \omega_2 t$$

BEAT PHENOMENON





$$u_1(0) = u_0, \qquad u_2(0) = 0.$$

$$\dot{u}_1(0) = \dot{u}_2(0) = 0$$

(a) Obtain the natural frequencies and mode shapes of the system.
(b) Determine expressions for the motion of the two masses given the initial conditions.
(c) Let system parameters *k, k',* and *m* be such that f_1 = 5.0 Hz and f_2 = 5.5 Hz and let the initial displacement be u_0 = 1. Plot the respective responses of the two masses.

Solution: Natural Frequencies and Mode Shapes

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k+k' & -k' \\ -k' & k+k' \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} \cos \omega t$$ and let $k' = \delta k, \qquad \lambda = \frac{\omega^2 m}{k}$

$$\left[\begin{bmatrix} 1+\delta & -\delta \\ -\delta & 1+\delta \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\lambda^2 - 2\lambda(1+\delta) + (1+2\delta) = 0$$

$$\left.\begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix}\right\} = (1+\delta) \mp \delta = \left\{\begin{matrix} 1 \\ 1+2\delta \end{matrix}\right. \Longrightarrow \omega_1 = \sqrt{\frac{k}{m}}, \qquad \omega_2 = \sqrt{\frac{k}{m}(1+2\delta)}$$

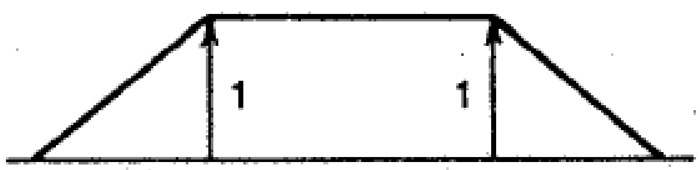
Solution: Natural Frequencies and Mode Shapes

$$\left[\begin{bmatrix} 1+\delta & -\delta \\ -\delta & 1+\delta \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

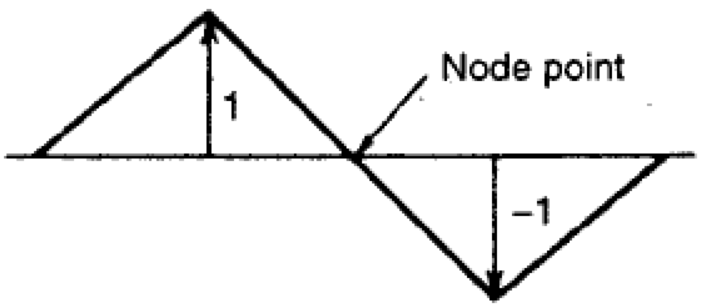
$$\beta_i \equiv \left(\frac{U_2}{U_1}\right)^{(i)} = \frac{1+\delta-\lambda_i}{\delta}$$

$$\beta_1 = 1, \qquad \beta_2 = -1$$

$$\phi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \qquad \phi_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$



Symmetric mode



Antisymmetric mode,

Solution: Free Vibration Response

$$u_1(t) = \frac{u_0}{2}(\cos\omega_1 t + \cos\omega_2 t)$$

$$u_2(t) = \frac{u_0}{2}(\cos\omega_1 t - \cos\omega_2 t)$$

$$\boxed{(1+\delta)^n \approx (1+n\delta),\ \text{if } \delta \ll 1} \quad \omega_1 = \sqrt{\frac{k}{m}}, \qquad \omega_2 = \sqrt{\frac{k}{m}(1+2\delta)} \approx (1+\delta)\sqrt{\frac{k}{m}}$$

$$\boxed{\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta} \qquad \boxed{\alpha = \frac{\omega_2+\omega_1}{2}t, \qquad \beta = \frac{\omega_2-\omega_1}{2}t}$$

$$u_1(t) = u_0\left(\cos\frac{\omega_2-\omega_1}{2}t\ \cos\frac{\omega_2+\omega_1}{2}t\right)$$

$$u_2(t) = u_0\left(\sin\frac{\omega_2-\omega_1}{2}t\ \sin\frac{\omega_2+\omega_1}{2}t\right)$$

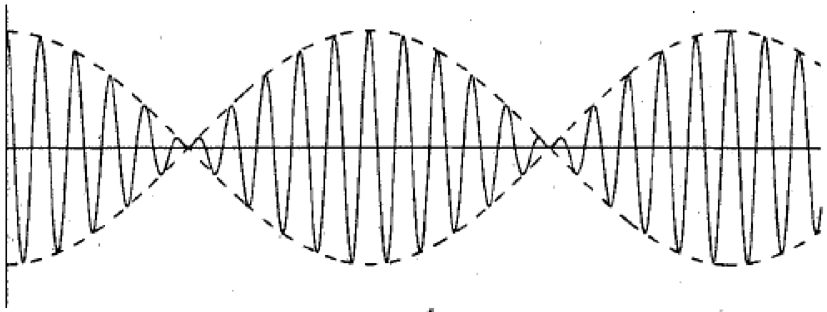
$$u_1(t) = \left(u_0\cos\frac{\omega_B t}{2}\right)\cos\omega_{avg}t$$

$$u_2(t) = \left(u_0\sin\frac{\omega_B t}{2}\right)\sin\omega_{avg}t$$

Solution: Frequency Assignment

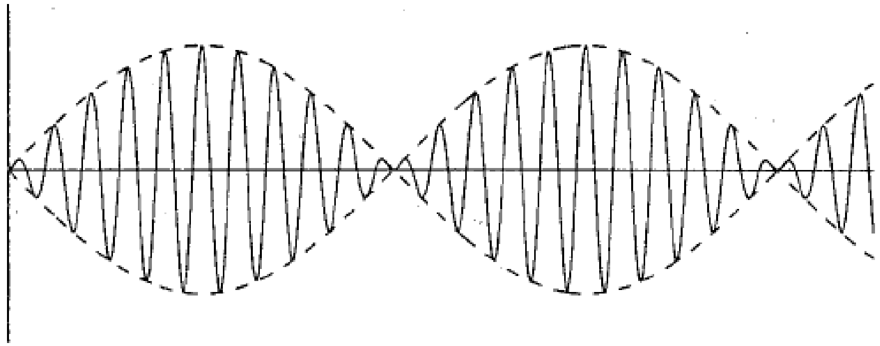
$$f_1 = \frac{\omega_1}{2\pi} = 5.00 \text{ Hz}, \qquad f_2 = \frac{\omega_2}{2\pi} = 5.50 \text{ Hz}$$

$$\omega_B = \omega_2 - \omega_1 = \pi \text{ rad/sec}, \qquad \omega_{avg} = \frac{\omega_2 + \omega_1}{2} = 10.5\,\pi \text{ rad/sec}$$



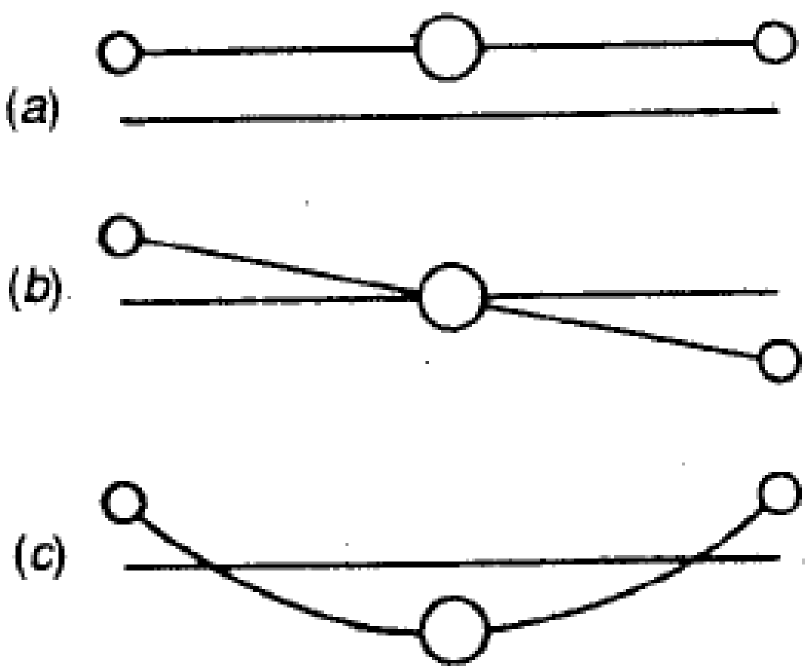
$$(u_1)_{env} = u_0 \cos\frac{\omega_B t}{2}$$

$$(u_2)_{env} = u_0 \sin\frac{\omega_B t}{2}$$



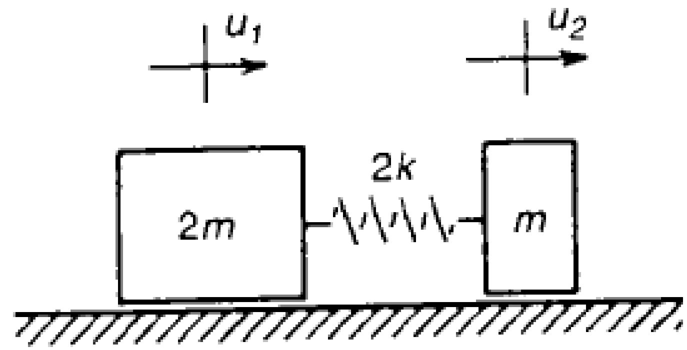
FREE VIBRATION OF SYSTEMS WITH RIGID-BODY MODES

Rigid-body modes have a corresponding natural frequency of zero (*semidefinite eigenvalue problem)*.



(a) mode I: translation (plunge) rigid-body mode;
(b) mode 2: rotation (roll) rigid-body mode;
(c) mode 3: flexible mode.

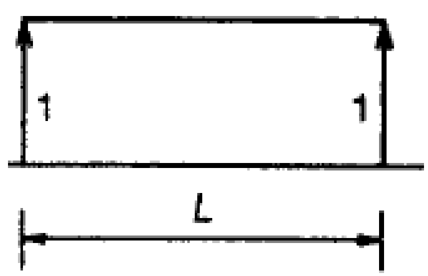
FREE VIBRATION OF SYSTEMS WITH RIGID-BODY MODES

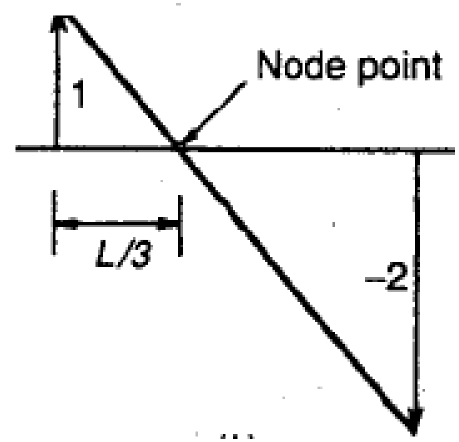


$$\begin{bmatrix} 2k - 2m\,\omega_i^2 & -2k \\ -2k & 2k - m\,\omega_i^2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\omega_1 = 0,\ \boldsymbol{\phi}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\omega_2 = \left(\frac{3k}{m}\right)^{1/2};\ \boldsymbol{\phi}_2 = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$$



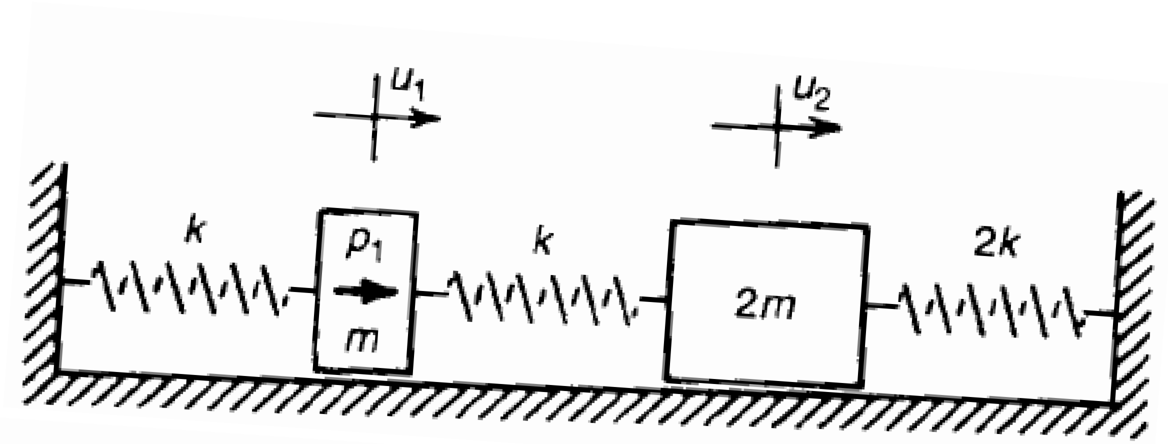


INTRODUCTION TO MODE SUPERPOSITION: FREQUENCY RESPONSE

This section considers the response of a system to harmonic excitation.

And provide an introduction to the *mode-superposition method* for solving for the dynamic response of MDOF systems.

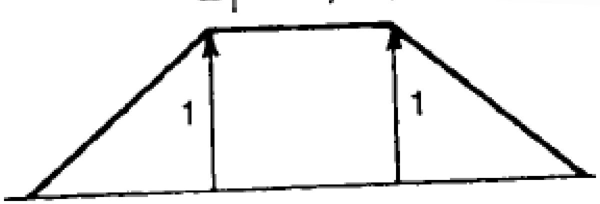
Response to Harmonic Excitation



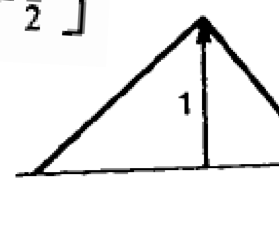
$$m\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{Bmatrix}\ddot{u}_1\\ \ddot{u}_2\end{Bmatrix}+k\begin{bmatrix}2 & -1\\ -1 & 3\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}=\begin{Bmatrix}P_1\\ 0\end{Bmatrix}\cos\Omega t$$

$$\mathbf{\Phi}=[\phi_1\,\phi_2]=\begin{bmatrix}\phi_{11} & \phi_{12}\\ \phi_{21} & \phi_{22}\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & -\frac{1}{2}\end{bmatrix}$$

$\omega_1^2 = k/m,$



$\omega_2^2 = \frac{5}{2}(k/m)$



Response to Harmonic Excitation

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix}$$

$$\boxed{\mathbf{u}(t) = \mathbf{\Phi}\,\boldsymbol{\eta}(t)}$$

$$\mathbf{\Phi}^T[\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{p}(t)]$$

$$\boxed{\mathcal{M}\ddot{\boldsymbol{\eta}} + \mathcal{K}\boldsymbol{\eta} = \boldsymbol{p}(t)}$$

$$\boxed{\mathcal{M} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}, \quad \mathcal{K} = \mathbf{\Phi}^{\mathrm{T}}K\,\mathbf{\Phi}, \quad \boldsymbol{p}(t) = \mathbf{\Phi}^{\mathrm{T}}\mathbf{p}(t)}$$

$$\mathcal{M} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} = m \begin{bmatrix} 3 & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$$

$$\mathcal{K} = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2k & -k \\ -k & 3k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} = k \begin{bmatrix} 3 & 0 \\ 0 & \frac{15}{4} \end{bmatrix}$$

$$\boldsymbol{p}(t) = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} P_1 \\ 0 \end{Bmatrix} \cos\Omega t = \begin{Bmatrix} P_1 \\ P_1 \end{Bmatrix} \cos\Omega t$$

Response to Harmonic Excitation (*modal response)*

$$3m\ddot{\eta}_1 + 3k\eta_1 = P_1 \cos \Omega t$$

$$\tfrac{3}{2}m\ddot{\eta}_2 + \tfrac{15}{4}k\eta_2 = P_1 \cos \Omega t$$

$$\eta_1 = Y_1 \cos \Omega t \qquad Y_1 = \frac{P_1}{3k - 3m\,\Omega^2} = \frac{(1/3k)\,P_1}{1 - (\Omega/\omega_1)^2}$$

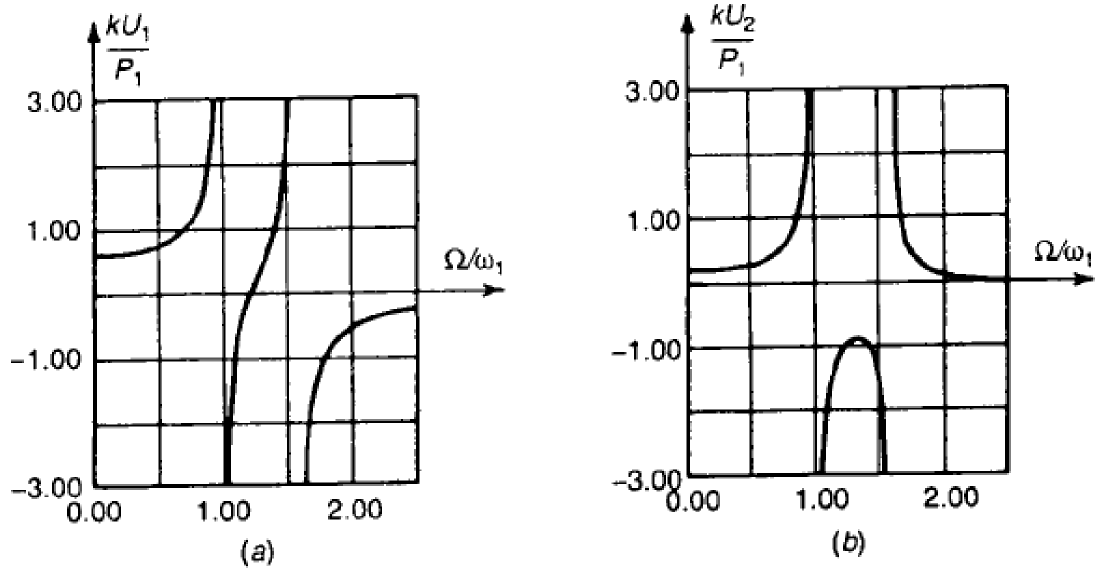
$$\eta_2 = Y_2 \cos \Omega t \qquad Y_2 = \frac{P_1}{\frac{15}{4}k - \frac{3}{2}m\Omega^2} = \frac{(4/15k)\,P_1}{1 - (\Omega/\omega_2)^2}$$

Response to Harmonic Excitation *(Physical coordinates response)*

$$u_1 = U_1 \cos \Omega t, \qquad u_2 = U_2 \cos \Omega t$$

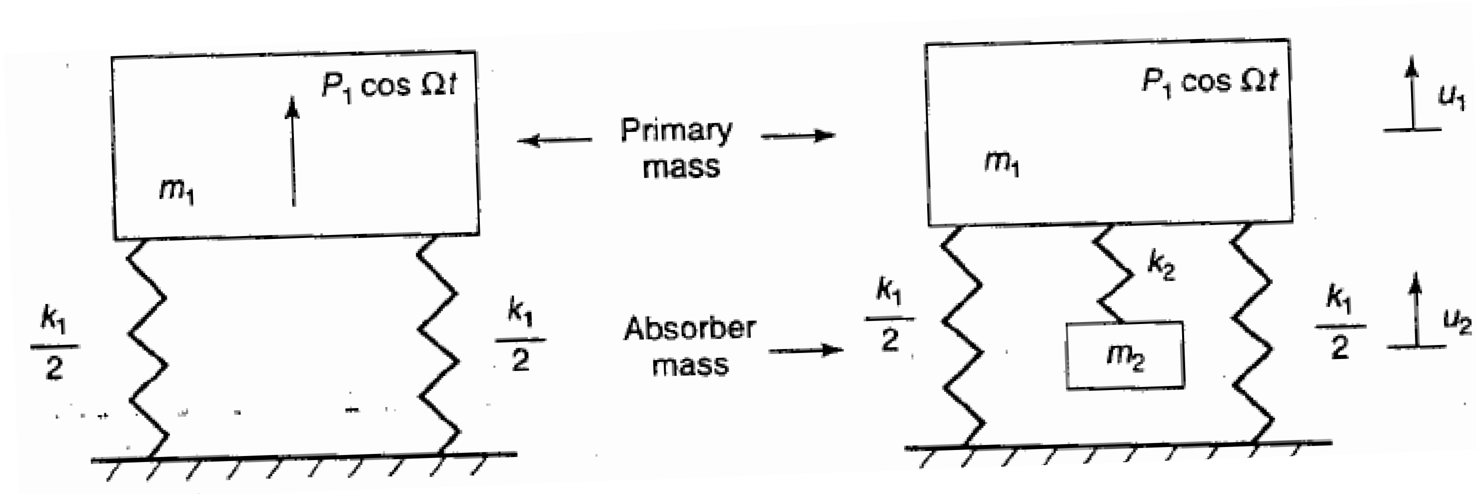
$$U_1 = \frac{P_1/3k}{1-(\Omega/\omega_1)^2} + \frac{4P_1/15k}{1-(\Omega/\omega_2)^2}$$

$$U_2 = \frac{P_1/3k}{1-(\Omega/\omega_1)^2} - \frac{1}{2}\left[\frac{4P_1/15k}{1-(\Omega/\omega_2)^2}\right]$$



Frequency-response functions for (*a*) mass 1 and (*b*) mass 2.

UNDAMPED VIBRATION ABSORBER



$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \end{Bmatrix} \cos \Omega t$$

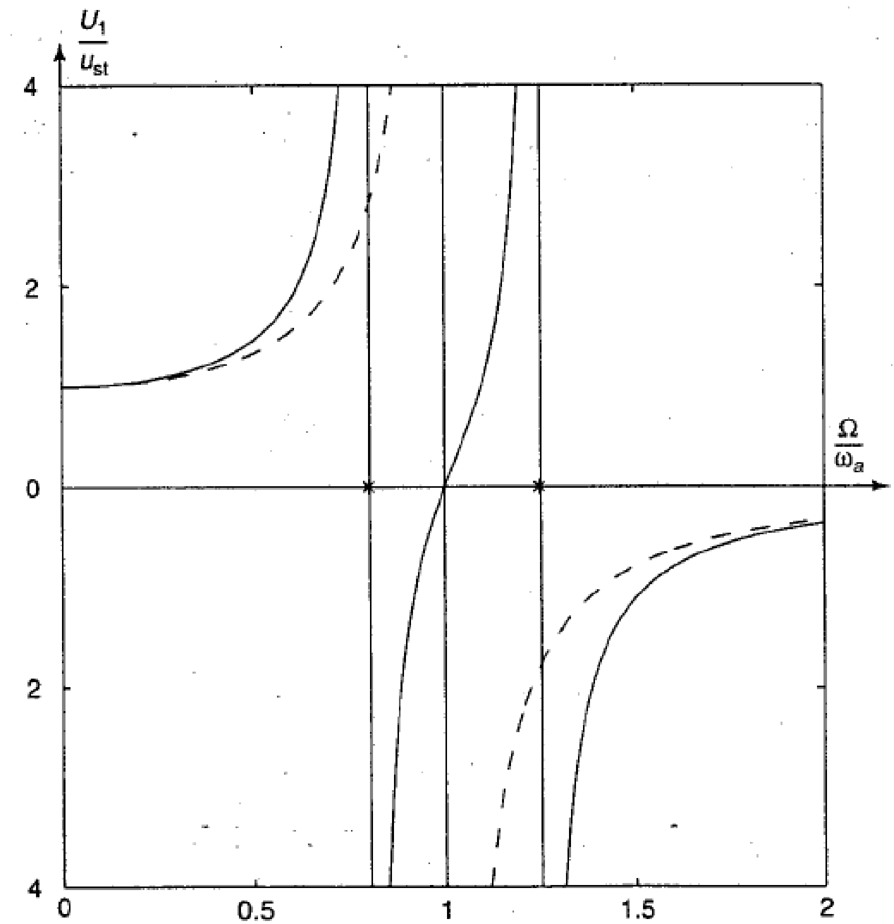
$$u_1(t) = U_1 \cos \Omega t, \qquad u_2(t) = U_2 \cos \Omega t$$

UNDAMPED VIBRATION ABSORBER

$$\begin{bmatrix} k_1 + k_2 - \Omega^2 m_1 & -k_2 \\ -k_2 & k_2 - \Omega^2 m_2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \end{Bmatrix}$$

$$U_1(\Omega) = \frac{(k_2 - \Omega^2 m_2) P_1}{(k_1 + k_2 - \Omega^2 m_1)(k_2 - \Omega^2 m_2) - k_2^2}$$

$$U_2(\Omega) = \frac{k_2 P_1}{(k_1 + k_2 - \Omega^2 m_1)(k_2 - \Omega^2 m_2) - k_2^2}$$



Vibration of Undamped 2-DOFSystems

The natural frequencies and natural modes of a system.

Expressions for the response to initial conditions.

> The beat phenomenon.

The natural frequencies/natural modes of systems.

The modal matrix, the modal stiffness matrix, and the modal mass matrix.

Expressions for the steady-state frequency response in principal coordinates and in physical coordinates.

> An undamped vibration absorber.

Structural Dynamics

Lecture Six: Vibration Properties of MDOF Systems (Chapter 10)

By: **H. Ahmadian**

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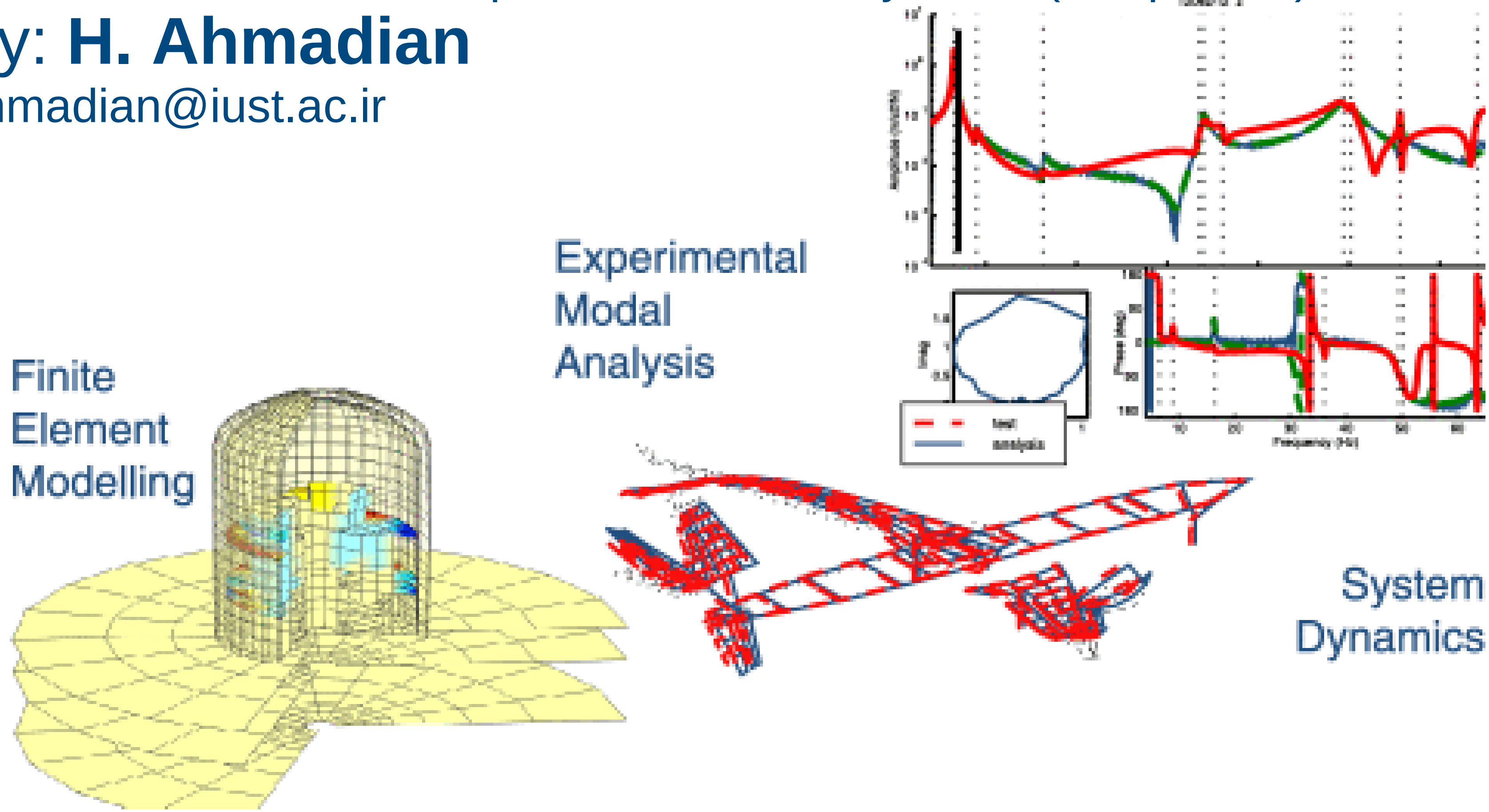


School of Mechanical Engineering

Iran University of Science and Technology

Vibration Properties of MDOF Systems: Modes, Frequencies, and Damping

1. Some Properties of Natural Frequencies and Natural Modes of Undamped MDOF Systems
2. Model Reduction: Rayleigh, Rayleigh-Ritz, and Assumed-Modes Methods
3. Uncoupled Damping in MDOF Systems
4. Structures with Arbitrary Viscous Damping: Complex Modes
5. Natural Frequencies and Mode Shapes of Damped Structures with Rigid-Body Modes

Properties of K and M

Stiffness matrix K and mass matrix M are related to strain energy and kinetic energy by the quadratic forms:

$$\mathcal{V} = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u}, \qquad \mathcal{T} = \frac{1}{2}\dot{\mathbf{u}}^T\mathbf{M}\dot{\mathbf{u}}$$

$$\mathbf{K}^T = \mathbf{K} \text{ and } \mathbf{M}^T = \mathbf{M}$$

For most structures **K** and **M** are *positive definite matrices*

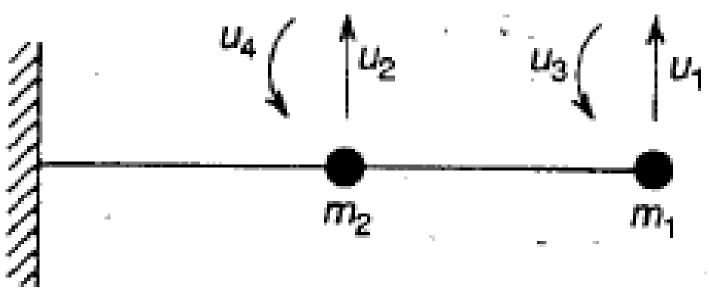
- For any arbitrary displacement of a system with positive definite **K** from its undeformed configuration, the strain energy will be positive.
- For any arbitrary velocity distribution of a system with positive definite **M**, a positive kinetic energy will result.

Exceptions

K of systems that have rigid-body freedom:

- **K** is said to be *positive semidefinite,*
- *Strain energy* can be either zero (for rigid-body motion) or greater than zero (for motion resulting in deformation)
- **K** is a *singular matrix.*

M of systems with degrees of freedom that have no associated inertia:



$$\mathbf{M} \doteq \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigensolution

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}$$

$$\mathbf{u}(t) = \mathbf{U}\cos(\omega t - \alpha)$$

$$[\mathbf{K} - \omega^2\mathbf{M}]\mathbf{U} = \mathbf{0}$$

Characteristic equation

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$$

$$0 \le \omega_1^2 \le \omega_2^2 \le \cdots \le \omega_r^2 \le \cdots \le \omega_N^2$$

$$\mathbf{U}_r = \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_N \end{Bmatrix}_r, \qquad r = 1, 2, \ldots, N$$

Scaling (Normalizing) the Modes

If the value of one of the *N* elements of a natural mode vector $\mathbf{U}_r$ is assigned a specified value, the remaining *N* - 1 elements are determined uniquely.

- Thus, we say that the *mode shape* is determined uniquely, but not the mode's amplitude.

There are three commonly employed procedures for normalizing modes:

- Scale the rth mode so that $(\phi_i)_r = 1$ a specified coordinate i.
- Scale the rth mode so that the maximum displacement is unity.
- Scale the rth mode so that its generalized mass, or *modal mass,* $\boldsymbol{\phi}_r^T \mathbf{M} \boldsymbol{\phi}_r$ is unity.

Mode Shapes: Distinct Frequencies

$$\mathbf{D}(\omega_r) \triangleq \mathbf{K} - \omega_r^2 \mathbf{M}$$

$$\begin{bmatrix} D_{aa}(\omega_r) & \mathbf{D}_{ab}(\omega_r) \\ \mathbf{D}_{ba}(\omega_r) & \mathbf{D}_{bb}(\omega_r) \end{bmatrix} \begin{Bmatrix} 1 \\ \boldsymbol{\phi}_b \end{Bmatrix}_r = \begin{Bmatrix} 0 \\ \mathbf{0}_b \end{Bmatrix} \qquad \{\boldsymbol{\phi}_b\}_r \equiv \begin{Bmatrix} \phi_2 \\ \phi_3 \\ \vdots \\ \phi_N \end{Bmatrix}_r$$

$$\{\boldsymbol{\phi}_b\}_r = -[\mathbf{D}_{bb}(\omega_r)]^{-1}\mathbf{D}_{ba}(\omega_r)$$

Orthogonality

$$(\boldsymbol{\phi}_s^{\mathrm{T}}\mathbf{K}\boldsymbol{\phi}_r) - \omega_r^2(\boldsymbol{\phi}_s^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}_r) = 0$$

$$(\boldsymbol{\phi}_r^{\mathrm{T}}\mathbf{K}\boldsymbol{\phi}_s) - \omega_s^2(\boldsymbol{\phi}_r^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}_s) = 0$$

$$(\omega_s^2 - \omega_r^2)(\boldsymbol{\phi}_s^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}_r) = 0$$

$$\boxed{\boldsymbol{\phi}_s^{\mathrm{T}}\mathbf{M}\boldsymbol{\phi}_r = 0 \qquad \text{if } \omega_r \neq \omega_s}$$

$$\boxed{\boldsymbol{\phi}_s^{\mathrm{T}}\mathbf{K}\boldsymbol{\phi}_r = 0 \qquad \text{if } \omega_r \neq \omega_s}$$

Mode Shapes: Repeated Frequencies

If the eigenvalue is repeated *p* times, there will be *p linearly independent eigenvectors* associated with this repeated eigenvalue.

$$\begin{bmatrix} \underset{p\times p}{\mathbf{D}_{aa}(\omega_r)} & \underset{p\times(N-p)}{\mathbf{D}_{ab}(\omega_r)} \\ \underset{(N-p)\times p}{\mathbf{D}_{ba}(\omega_r)} & \underset{(N-p)\times(N-p)}{\mathbf{D}_{bb}(\omega_r)} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\phi}_a \\ \boldsymbol{\phi}_b \end{Bmatrix}_r = \begin{Bmatrix} \mathbf{0}_a \\ \mathbf{0}_b \end{Bmatrix} \qquad \{\boldsymbol{\phi}_a\} \equiv \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{Bmatrix}_r, \qquad \{\boldsymbol{\phi}_b\} \equiv \begin{Bmatrix} \phi_{p+1} \\ \phi_{p+2} \\ \vdots \\ \phi_N \end{Bmatrix}_r$$

$$\boxed{\{\boldsymbol{\phi}_b\}_r = -[\mathbf{D}_{bb}(\omega_r)]^{-1}\mathbf{D}_{ba}(\omega_r)\{\boldsymbol{\phi}_a\}_r}$$

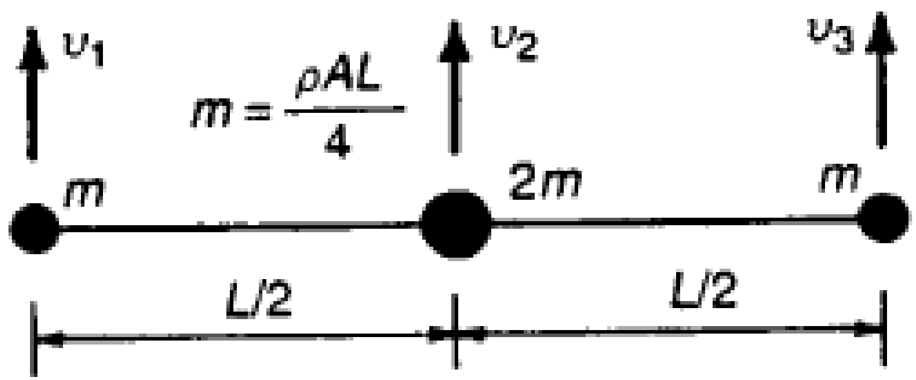
$$\{\boldsymbol{\phi}_a\}_r \equiv \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_p \end{Bmatrix}_r = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}, \quad \{\boldsymbol{\phi}_a\}_{(r+1)} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}, \quad \ldots, \quad \{\boldsymbol{\phi}_a\}_{(r+p-1)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{Bmatrix}$$

Example:

(a) Solve for the three natural frequencies of this system.

(b) Solve for the three normal modes of the system.

(c) Evaluate the orthogonality relationships.



$$\frac{\rho A L}{4}\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{v}_1\\ \ddot{v}_2\\ \ddot{v}_3\end{Bmatrix}+\frac{12EI}{L^3}\begin{bmatrix}1 & -2 & 1\\ -2 & 4 & -2\\ 1 & -2 & 1\end{bmatrix}\begin{Bmatrix}v_1\\ v_2\\ v_3\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}$$

Natural Frequencies:

$$[\mathbf{K} - \omega^2 \mathbf{M}]\boldsymbol{\phi} = \mathbf{0}$$

$$\mathbf{D}(\lambda)\boldsymbol{\phi} = \begin{bmatrix} 1-\lambda & -2 & 1 \\ -2 & 2(2-\lambda) & -2 \\ 1 & -2 & 1-\lambda \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \lambda = \omega^2 \frac{\rho A L^4}{48 EI}$$

$$\lambda^2(\lambda - 4) = 0$$

$$\lambda_1 = \lambda_2 = 0, \qquad \lambda_3 = 4$$

$$\omega_1^2 = \omega_2^2 = 0, \qquad \omega_3^2 = 4\frac{48EI}{\rho A L^4}$$

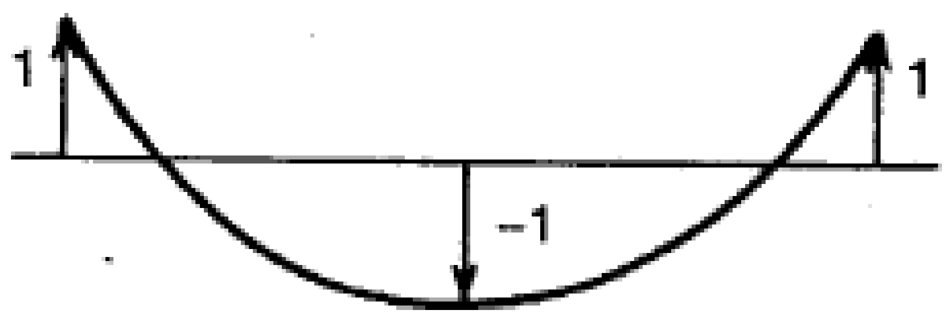
Normal Modes:

$$\lambda_3 = 4 \quad \Longrightarrow \quad \begin{bmatrix} D_{aa}(\lambda_3) & \mathbf{D}_{ab}(\lambda_3) \\ \mathbf{D}_{ba}(\lambda_3) & \mathbf{D}_{bb}(\lambda_3) \end{bmatrix} \begin{Bmatrix} 1 \\ \boldsymbol{\phi}_b \end{Bmatrix}_3 = \left[\begin{array}{c|cc} -3 & -2 & 1 \\ \hline -2 & -4 & -2 \\ 1 & -2 & -3 \end{array}\right] \begin{Bmatrix} 1 \\ \hline \phi_2 \\ \phi_3 \end{Bmatrix}_3$$

$$\{\boldsymbol{\phi}_b\}_3 = \begin{Bmatrix} \phi_2 \\ \phi_3 \end{Bmatrix}_3 = -[\mathbf{D}_{bb}(\lambda_3)]^{-1}\mathbf{D}_{ba}(\lambda_3)$$

$$= -\frac{1}{8}\begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix} \begin{Bmatrix} -2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\boldsymbol{\phi}_3 = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$



Normal Modes:

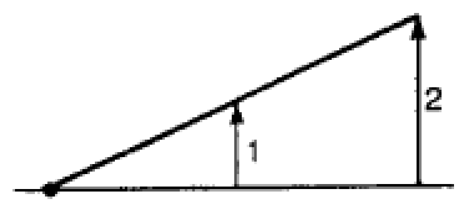
$$\lambda = 0 \Longrightarrow \quad \mathbf{D}(0) = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{D}_{aa}(\lambda_3) & \mathbf{D}_{ab}(\lambda_3) \\ \mathbf{D}_{ba}(\lambda_3) & D_{bb}(\lambda_3) \end{bmatrix} \begin{Bmatrix} \boldsymbol{\phi}_a \\ \phi_b \end{Bmatrix} = \left[\begin{array}{cc|c} 1 & -2 & 1 \\ -2 & 4 & -2 \\ \hline 1 & -2 & 1 \end{array}\right] \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \hline \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$(\phi_3)_r = -1[\,1 \quad -2\,] \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix}_r, \qquad r = 1, 2$$

$$(\phi_3)_1 = [\,-1 \quad 2\,] \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = -1 \qquad\qquad (\phi_3)_2 = [\,-1 \quad 2\,] \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = 2$$

$$\boldsymbol{\phi}_1 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix},$$



$$\boldsymbol{\phi}_2 = \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix}$$

Orthogonality Relationships:

$$\boldsymbol{\phi}_1^T \mathbf{M} \boldsymbol{\phi}_2 = [\,1 \;\; 0 \;\; -1\,] \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix} = -2m \neq 0$$

$$\boldsymbol{\phi}_1^T \mathbf{M} \boldsymbol{\phi}_3 = [\,1 \;\; 0 \;\; -1\,] \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix} = 0$$

Orthogonality Relationships:

It is possible to use a set of *p* linearly independent modes to create a set of *p* modes that are orthogonal (*Gram-Schmidt Procedure*):

$$\hat{\boldsymbol{\phi}}_2 = \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2,$$

$$\boldsymbol{\phi}_1^{\mathrm{T}}\mathbf{M}\hat{\boldsymbol{\phi}}_2 = [\,1 \quad 0 \quad -1\,]\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 0$$

Modal Matrix and Eigenvalue Matrix

$$\boxed{\mathbf{\Phi} \equiv [\boldsymbol{\phi}_1 \ \boldsymbol{\phi}_2 \ \cdots \ \boldsymbol{\phi}_N]}$$

$$\mathbf{K\Phi} = \mathbf{M\Phi\Lambda}$$

$$\boxed{\mathbf{\Lambda} \equiv \mathrm{diag}(\omega_1^2, \omega_2^2, \ldots, \omega_N^2)}$$

Generalized Mass Stiffness Matrices

$$\boxed{\mathcal{M} = \mathbf{\Phi}^T\mathbf{M\Phi} = \mathrm{diag}(M_1, M_2, \ldots, M_N)}$$

$$\boxed{\mathcal{K} = \mathbf{\Phi}^T\mathbf{K\Phi} = \mathrm{diag}(K_1, K_2, \ldots, K_N)}$$

$$\mathbf{\Phi}^T\mathbf{M\Phi} = \mathbf{I} \qquad \mathbf{\Phi}^T\mathbf{K\Phi} = \mathbf{\Lambda}$$

Mode Superposition Employing Modes of the Undamped Structure

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{p}(t)$$

Mode superposition using real modes of the undamped system	or	***Mode superposition*** using complex modes of the damped system	or	***Direct integration*** of the coupled equations of motion

$$\boxed{\mathbf{u}(t) = \sum_{r=1}^{N} \boldsymbol{\phi}_r \eta_r(t) = \boldsymbol{\Phi}\boldsymbol{\eta}(t)}$$

$$\boxed{\mathcal{M}\ddot{\boldsymbol{\eta}} + \mathcal{C}\dot{\boldsymbol{\eta}} + \mathcal{K}\boldsymbol{\eta} = \boldsymbol{\Phi}^T\mathbf{p}(t)}$$

$$\begin{aligned}
\mathcal{M} &= \boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi} = \text{modal mass matrix (diag.)}\\
\mathcal{C} &= \boldsymbol{\Phi}^T\mathbf{C}\boldsymbol{\Phi} = \text{generalized damping matrix}\\
\mathcal{K} &= \boldsymbol{\Phi}^T\mathbf{K}\boldsymbol{\Phi} = \text{modal stiffness matrix (diag.)}\\
& \quad\ \boldsymbol{\Phi}^T\mathbf{p}(t) = \text{modal force vector}
\end{aligned}$$

Structural Dynamics

Lecture Seven: Vibration Properties of MDOF Systems (Chapter 10)

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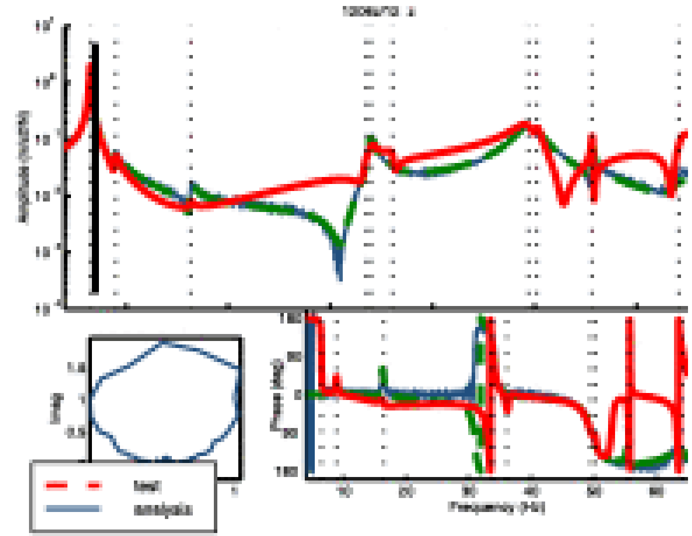
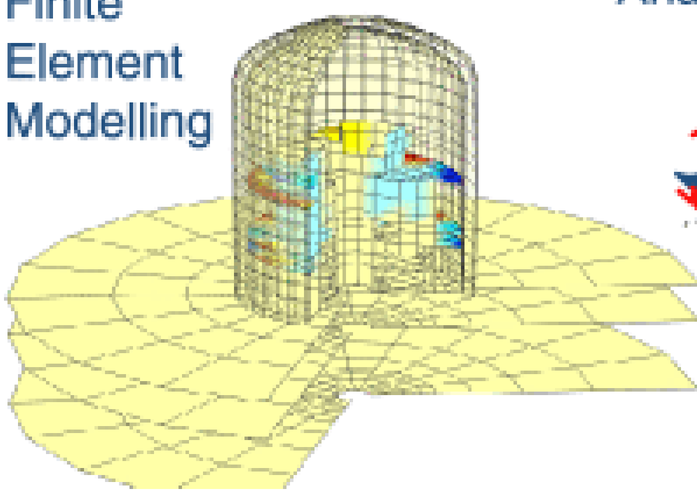
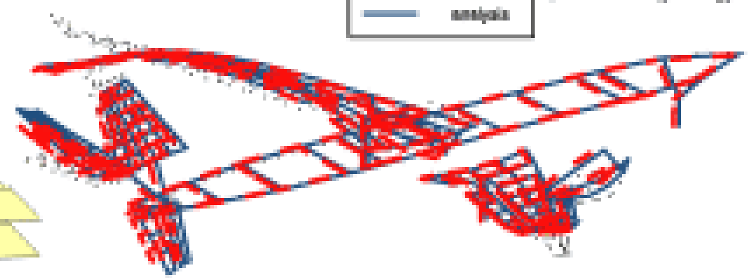


Experimental
Modal
Analysis

Finite
Element
Modelling





System
Dynamics

School of Mechanical Engineering
Iran University of Science and Technology

Vibration Properties of MDOF Systems: Modes, Frequencies, and Damping

1. Some Properties of Natural Frequencies and Natural Modes of Undamped MDOF Systems
2. Model Reduction: Rayleigh, Rayleigh-Ritz, and Assumed-Modes Methods
3. Uncoupled Damping in MDOF Systems
4. Structures with Arbitrary Viscous Damping: Complex Modes
5. Natural Frequencies and Mode Shapes of Damped Structures with Rigid-Body Modes

Rayleigh Quotient

$$\boxed{\omega_R^2 \equiv \mathcal{R}(\mathbf{v}) = \frac{\mathbf{v}^{\mathrm{T}}\mathbf{K}\mathbf{v}}{\mathbf{v}^{\mathrm{T}}\mathbf{M}\mathbf{v}}}$$

$$\mathbf{v} = \sum_{r=1}^{N} c_r \boldsymbol{\phi}_r$$

$$\mathcal{R}(\mathbf{v}) = \frac{\omega_1^2 c_1^2 + \omega_2^2 c_2^2 + \cdots + \omega_N^2 c_N^2}{c_1^2 + c_2^2 + \cdots + c_N^2}$$

Rayleigh Quotient
Error Analysis

$$\mathcal{R}(\mathbf{v}) = \frac{\omega_1^2 c_1^2 + \omega_2^2 c_2^2 + \cdots + \omega_N^2 c_N^2}{c_1^2 + c_2^2 + \cdots + c_N^2}$$

$$\omega_1^2 \leq \mathcal{R}(\mathbf{v}) \leq \omega_N^2$$

$$\mathcal{R}(\mathbf{v}) = \omega_1^2 \frac{1 + (c_2/c_1)^2(\omega_2/\omega_1)^2 + \cdots + (c_N/c_1)^2(\omega_N/\omega_1)^2}{1 + (c_2/c_1)^2 + \cdots + (c_N/c_1)^2}$$

$$\mathcal{R}(\mathbf{v}) \geq \omega_1^2$$

Example:

$$M = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad K = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$$\mathbf{F} = c[m_1 \quad m_2 \quad m_3]^T = [1 \quad 1 \quad 2]^T$$

$$\mathbf{u} = \frac{1}{k}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{k}\begin{bmatrix} 4 \\ 7 \\ 8 \end{bmatrix}$$

Example:

$$R = \omega^2 = \frac{\mathbf{u}^T K \mathbf{u}}{\mathbf{u}^T M \mathbf{u}} = \frac{27k}{193m} = 0.1399\frac{k}{m}$$

$$\omega = 0.3740\sqrt{\frac{k}{m}} \qquad \omega_1 = 0.3731\sqrt{\frac{k}{m}}$$

Exact solution

$$\frac{\omega - \omega_1}{\omega_1} = \frac{0.3740 - 0.3731}{0.3731} = 0.002412 = 0.2412\%$$

Rayleigh-Ritz Method for MDOF Systems

$$\mathbf{u}(t) = \mathbf{v}\cos(\omega t - \alpha)$$

Preselected linearly independent assumed-mode vectors.

$$\mathbf{v} = \sum_{i=1}^{\hat{N}} \boldsymbol{\psi}_i \hat{v}_i = \hat{\boldsymbol{\Psi}}\hat{\mathbf{v}} \qquad (\hat{N} < N)$$

$$\mathcal{V}_{\max} = \mathcal{T}_{\max} \longrightarrow \mathcal{R}(\mathbf{v}) \equiv \hat{\omega}^2 = \frac{\hat{\mathbf{v}}^T\hat{\mathbf{K}}\hat{\mathbf{v}}}{\hat{\mathbf{v}}^T\hat{\mathbf{M}}\hat{\mathbf{v}}} = \frac{\sum_{i=1}^{\hat{N}}\sum_{j=1}^{\hat{N}}\hat{v}_i\hat{v}_j\hat{k}_{ij}}{\sum_{i=1}^{\hat{N}}\sum_{j=1}^{\hat{N}}\hat{v}_i\hat{v}_j\hat{m}_{ij}}$$

$$\hat{\mathbf{K}} = \hat{\boldsymbol{\Psi}}^T\mathbf{K}\hat{\boldsymbol{\Psi}}, \qquad \hat{\mathbf{M}} = \hat{\boldsymbol{\Psi}}^T\mathbf{M}\hat{\boldsymbol{\Psi}}$$

Rayleigh-Ritz Method for MDOF Systems

Ritz proposal: the coefficients $\widehat{v}_i$ be chosen to make $\mathcal{R}(\mathbf{v})$ stationary $\dfrac{\partial \mathcal{R}(\mathbf{v})}{\partial \widehat{v}_i} = 0, \qquad i = 1, 2, \ldots, \widehat{N}$

$$\mathcal{R}(\mathbf{v}) \equiv \widehat{\omega}^2 = \frac{\widehat{\mathbf{v}}^T \widehat{\mathbf{K}} \widehat{\mathbf{v}}}{\widehat{\mathbf{v}}^T \widehat{\mathbf{M}} \widehat{\mathbf{v}}} = \frac{\sum_{i=1}^{\widehat{N}} \sum_{j=1}^{\widehat{N}} \widehat{v}_i \widehat{v}_j \widehat{k}_{ij}}{\sum_{i=1}^{\widehat{N}} \sum_{j=1}^{\widehat{N}} \widehat{v}_i \widehat{v}_j \widehat{m}_{ij}}$$

$$\sum_{j=1}^{\widehat{N}} (\widehat{k}_{ij} - \widehat{\omega}^2 \widehat{m}_{ij}) \widehat{v}_j = 0, \qquad i = 1, 2, \ldots, \widehat{N}$$

$$\boxed{[\widehat{\mathbf{K}} - \widehat{\omega}^2 \widehat{\mathbf{M}}]\, \widehat{\mathbf{v}} = \mathbf{0}}$$

Assumed-Modes Method for Model Reduction of MDOF Systems

The Rayleigh-Ritz procedure is a model order-reduction method that applies specifically to free vibration.

This method can be considered to be special case of applying the *Assumed-Modes Method* to reduce an N-DOF system to a $\hat{N}$-DOF system by assuming that:

$$\boxed{\mathbf{u}(t) = \sum_{i=1}^{\hat{N}} \psi_i \hat{u}_i(t) = \hat{\boldsymbol{\Psi}}\,\hat{\mathbf{u}}(t)}$$

$$\hat{\mathbf{M}}\ddot{\hat{\mathbf{u}}} + \hat{\mathbf{C}}\dot{\hat{\mathbf{u}}} + \hat{\mathbf{K}}\hat{\mathbf{u}} = \hat{\mathbf{p}}(t)$$

Eigenvalue Separation Property

$$[\mathbf{K} - \lambda \mathbf{M}]\mathbf{v} = \mathbf{0}$$

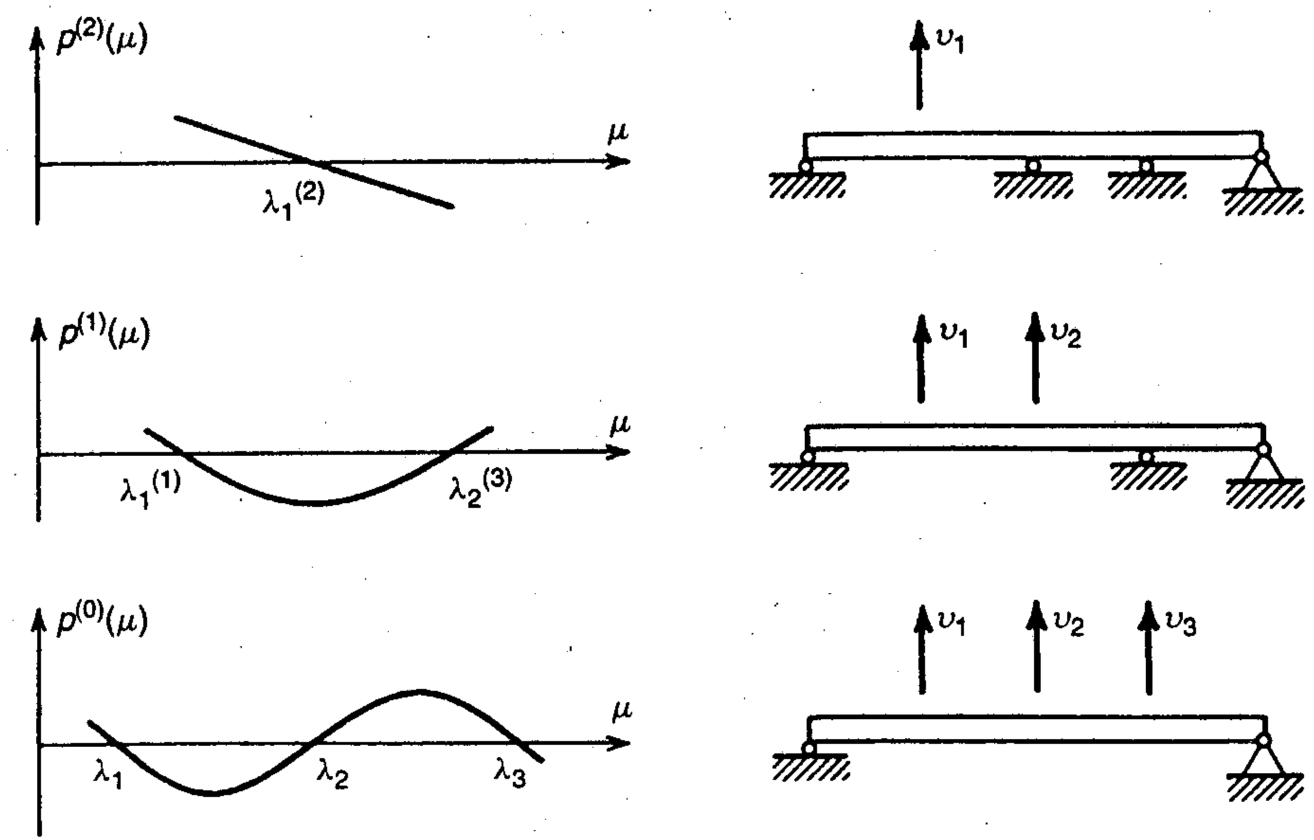
$$[\mathbf{K}^{(m)} - \lambda^{(m)}\mathbf{M}^{(m)}]\mathbf{v}^{(m)} = \mathbf{0}, \quad m = 0, 1, \ldots, N-1$$

$\mathbf{K}^{(m)}$ $\mathbf{M}^{(m)}$ obtained by deleting the last *m* rows and columns of **K** and **M**

$$\boxed{\lambda_1^{(m)} \le \lambda_1^{(m+1)} \le \lambda_2^{(m)} \le \lambda_2^{(m+1)} \le \cdots \le \lambda_{(N-m)}^{(m)} \qquad \text{for } m = 0, 1, 2, \ldots, N-2}$$

Eigenvalue Separation Property

$$\lambda_1^{(m)} \leq \lambda_1^{(m+1)} \leq \lambda_2^{(m)} \leq \lambda_2^{(m+1)} \leq \cdots \leq \lambda_{(N-m)}^{(m)} \qquad \text{for } m = 0, 1, 2, \ldots, N-2$$



Eigenvalue Separation Property

- The eigenvalue separation theorem can be employed directly to show the convergence properties of frequencies obtained by the Rayleigh-Ritz method.
- Each of the $\widehat{N} < N$ eigenvalues produced by a Rayleigh-Ritz approximation to an N-DOF system is an upper bound to the corresponding exact eigenvalue,
- The eigenvalues approach the exact values from above as the number of degrees of freedom, N increases.

Eigenvalue Separation Property

Table 10.1 Convergence Properties of Rayleigh–Ritz Frequencies

DOF $= \widehat{N} =$	1	2	3	$\cdots$	$N-1$	N
"Constraints" $= m =$	$N-1$	$N-2$	$N-3$	$\cdots$	1	0
First eigenvalue	$\lambda_1^{(N-1)} \geq$	$\lambda_1^{(N-2)} \geq$	$\lambda_1^{(N-3)} \geq$	$\cdots$	$\lambda_1^{(1)} \geq$	λ_1
Second eigenvalue		$\lambda_2^{(N-2)} \geq$	$\lambda_2^{(N-3)} \geq$	$\cdots$	$\lambda_2^{(1)} \geq$	λ_2
Third eigenvalue			$\lambda_3^{(N-3)} \geq$	$\cdots$	$\lambda_3^{(1)} \geq$	λ_3
$\vdots$					$\vdots$	
Nth eigenvalue						λ_N

UNCOUPLED DAMPING IN MDOF SYSTEMS

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{p}(t)$$

Equation of motion in principal coordinates:

$$\boldsymbol{M}\ddot{\boldsymbol{\eta}} + \boldsymbol{C}\dot{\boldsymbol{\eta}} + \boldsymbol{K}\boldsymbol{\eta} = \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{p}(t)$$

$$M = \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}\boldsymbol{\Phi} = \mathrm{diag}(M_r),$$

$$K = \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{K}\boldsymbol{\Phi} = \mathrm{diag}(K_r) = \mathrm{diag}(\omega_r^2 M_r)$$

$$C = \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{C}\boldsymbol{\Phi}$$

The *generalized damping matrix* is not diagonal.

UNCOUPLED DAMPING IN MDOF SYSTEMS: Raleigh Damping

$$\boxed{\mathbf{C} = a_0\mathbf{M} + a_1\mathbf{K}}$$

$$C = \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi} = \text{diag}(C_r) =$$

$$\text{diag}(a_0 + a_1\omega_r^2)M_r = \text{diag}(2\zeta_r\omega_r M_r)$$

$$\boxed{\zeta_r = \frac{1}{2}\left(\frac{a_0}{\omega_r} + a_1\omega_r\right)}$$

$$\boxed{M_r\ddot{\eta}_r + 2M_r\omega_r\zeta_r\dot{\eta}_r + \omega_r^2 M_r\eta_r = \boldsymbol{\phi}_r^T\mathbf{p}(t), \qquad r = 1, 2, \ldots, N}$$

UNCOUPLED DAMPING IN MDOF SYSTEMS: Modal Damping

$$\boxed{C = \Phi^T \mathbf{C}\, \Phi = \mathrm{diag}(C_r) = \mathrm{diag}(2\zeta_r\omega_r M_r)}$$

Typical values lie in the range $0.01 \le \zeta_r \le 0.1$.

$$\boxed{M_r\ddot{\eta}_r + 2M_r\omega_r\zeta_r\dot{\eta}_r + \omega_r^2 M_r\eta_r = \boldsymbol{\phi}_r^T \mathbf{p}(t), \qquad r = 1, 2, \ldots, N}$$

Damping Matrix C in Physical Coordinates

$$\mathbf{C} = \boldsymbol{\Phi}^{-T} C \, \boldsymbol{\Phi}^{-1}$$

$$\boldsymbol{\Phi}^{T}\mathbf{M}\,\boldsymbol{\Phi} = \mathbf{I}$$

$$\longrightarrow \quad \mathbf{C} = (\mathbf{M}\boldsymbol{\Phi})C\,(\boldsymbol{\Phi}^{T}\mathbf{M})$$

$$\boxed{\mathbf{C} = \sum_{r=1}^{N} 2\,\zeta_r \omega_r \,(\mathbf{M}\boldsymbol{\phi}_r)(\mathbf{M}\boldsymbol{\phi}_r)^{T}}$$

There will be no damping of those modes for which its damping ratio is set to zero.

Damping Matrix C for Augmented Modal Damping

$$\boxed{\mathbf{C} = a_1 \mathbf{K} + \sum_{r=1}^{N_c - 1} 2\hat{\zeta}_r \omega_r \mathbf{M}\boldsymbol{\phi}_r (\mathbf{M}\boldsymbol{\phi}_r)^{\mathrm{T}}}$$

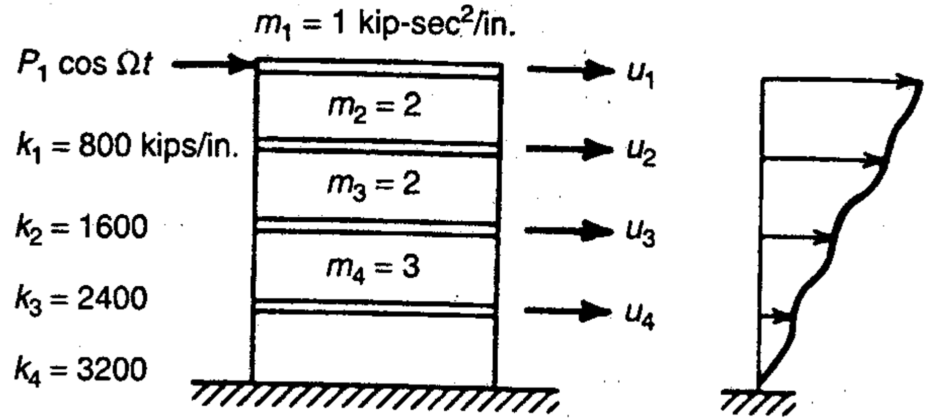
$$a_1 = \frac{2\zeta_{N_c}}{\omega_{N_c}}, \qquad \hat{\zeta}_r = \zeta_r - \zeta_{N_c}\frac{\omega_r}{\omega_{N_c}}$$

$$\zeta_r = \begin{cases} \text{specified value}, & r = 1, 2, \ldots, N_c \\ \zeta_{N_c}\dfrac{\omega_r}{\omega_{N_c}}, & r = N_c + 1, N_c + 2, \ldots, N \end{cases}$$

UNCOUPLED DAMPING IN MDOF SYSTEMS

Example 10.5 (a) Use Eq. 10.87 to define a physical damping matrix **C** for the four-story building in Fig. 1. Assign two damping factors: $\zeta_1 = \zeta_2 = 0.01$. (b) Determine the resulting damping ratios ζ_3 and ζ_4.

$$\mathbf{K} = 800\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 7 \end{bmatrix}$$



$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad \phi_1 = \begin{Bmatrix} 1.00000 \\ 0.77910 \\ 0.49655 \\ 0.23506 \end{Bmatrix} \quad \omega = \begin{Bmatrix} 13.294 \\ 29.660 \\ 41.079 \\ 55.882 \end{Bmatrix} \text{rad/sec}$$

UNCOUPLED DAMPING IN MDOF SYSTEMS

$$\mathbf{C} = a_1\mathbf{K} + \frac{2\hat{\zeta}_1\omega_1}{M_1}(\mathbf{M}\boldsymbol{\phi}_1)(\mathbf{M}\boldsymbol{\phi}_1)^T$$

$$a_1 = \frac{2\zeta_2}{\omega_2}, \qquad \hat{\zeta}_1 = \zeta_1 - \zeta_2\frac{\omega_1}{\omega_2}$$

$$M_1 = \boldsymbol{\phi}_1^T\mathbf{M}\boldsymbol{\phi}_1 = 2.87288 \text{ kip-sec}^2/\text{in.}$$

$$\mathbf{C} = \begin{bmatrix} 0.59051 & -0.45988 & 0.05071 & 0.03601 \\ -0.45988 & 1.74233 & -0.99987 & 0.05611 \\ 0.05071 & -0.99987 & 2.74760 & -1.58258 \\ 0.03601 & 0.05611 & -1.58258 & 3.80153 \end{bmatrix} \text{kip-sec/in.}$$

UNCOUPLED DAMPING IN MDOF SYSTEMS

$$\zeta_r = \zeta_2 \frac{\omega_r}{\omega_2}, \qquad r = 3, 4$$

$$\zeta_3 = 0.01\left(\frac{41.079}{29.660}\right) = 0.0138$$

$$\zeta_4 = 0.01\left(\frac{55.882}{29.660}\right) = 0.0188$$

Structural Dynamics

Lecture Eight: Vibration Properties of MDOF Systems (Chapter 10)

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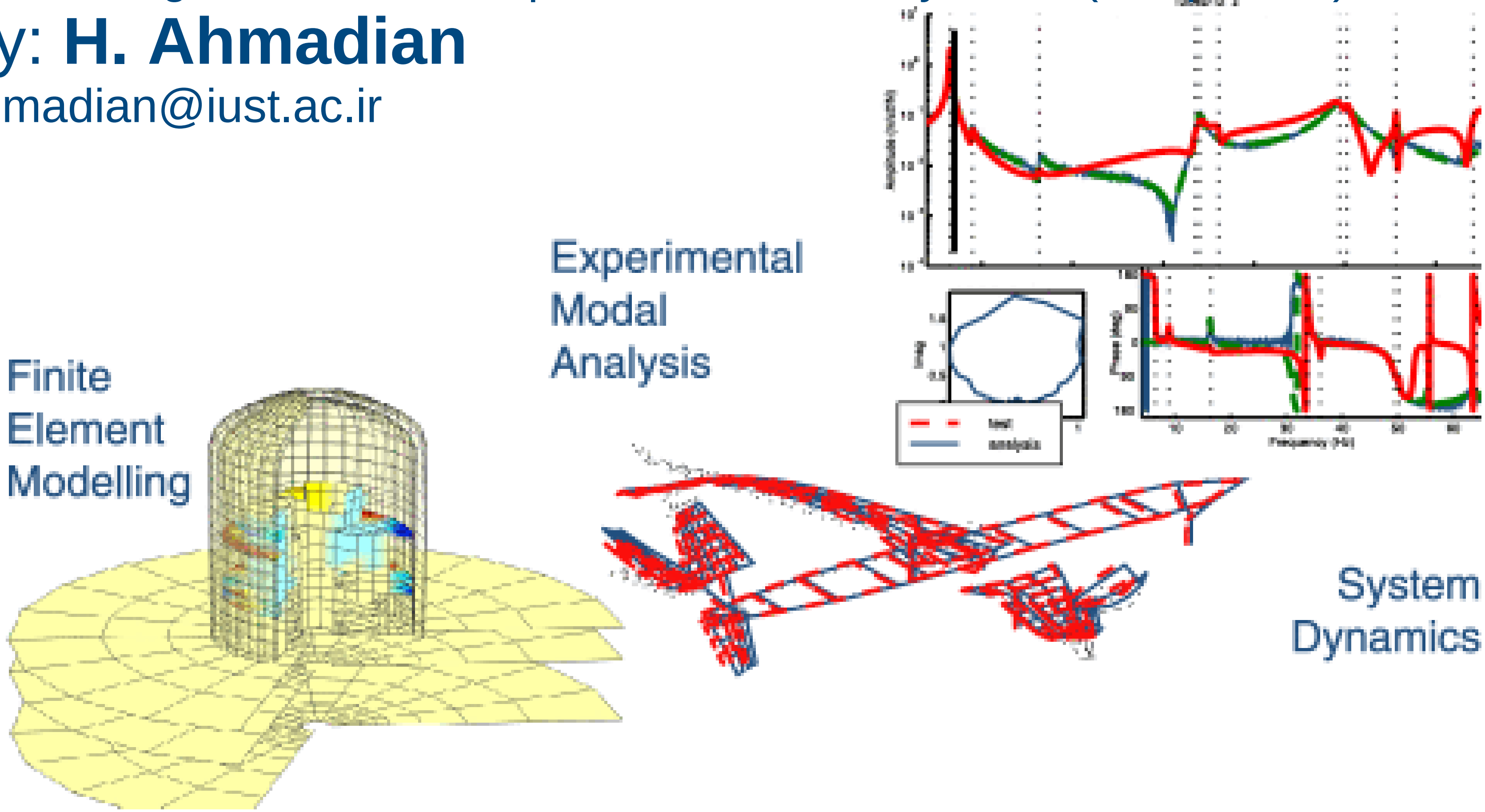

School of Mechanical Engineering
Iran University of Science and Technology

Vibration Properties of MDOF Systems: Modes, Frequencies, and Damping

1. Some Properties of Natural Frequencies and Natural Modes of Undamped MDOF Systems
2. Model Reduction: Rayleigh, Rayleigh-Ritz, and Assumed-Modes Methods
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4. Structures with Arbitrary Viscous Damping: Complex Modes
5. Natural Frequencies and Mode Shapes of Damped Structures with Rigid-Body Modes

STRUCTURES WITH ARBITRARY VISCOUS DAMPING: COMPLEX MODES

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{p}(t)$$

Equation of motion in principal coordinates:

$$\boldsymbol{M}\ddot{\boldsymbol{\eta}} + \boldsymbol{C}\dot{\boldsymbol{\eta}} + \boldsymbol{K}\boldsymbol{\eta} = \mathbf{\Phi}^T\mathbf{p}(t)$$

$$\boldsymbol{M} = \mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi} = \text{diag}(M_r),$$

$$\boldsymbol{K} = \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \text{diag}(K_r) = \text{diag}(\omega_r^2 M_r)$$

$$\boldsymbol{C} = \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi}$$

The *generalized damping matrix* is not diagonal.

State-Space Form of the Equations of Motion

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{p}(t)$$

$$\mathbf{z}(t) \equiv \begin{Bmatrix} \mathbf{u}(t) \\ \mathbf{v}(t) \end{Bmatrix}$$

$$\boxed{\mathbf{A}\dot{\mathbf{z}}(t) + \mathbf{B}\mathbf{z}(t) = \mathbf{F}(t)}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \quad \mathbf{F}(t) = \begin{Bmatrix} \mathbf{p}(t) \\ \mathbf{0} \end{Bmatrix}$$

State-Space Eigenvalue Problem

$$\mathbf{A}\dot{\mathbf{z}}(t) + \mathbf{B}\mathbf{z}(t) = \mathbf{0}$$

$$\mathbf{z}(t) = \boldsymbol{\theta}\, e^{\lambda t} \equiv \begin{Bmatrix} \boldsymbol{\theta}_u \\ \boldsymbol{\theta}_v \end{Bmatrix} e^{\lambda t} = \begin{Bmatrix} \boldsymbol{\theta}_u \\ \lambda \boldsymbol{\theta}_u \end{Bmatrix} e^{\lambda t}$$

$$\boxed{[\lambda \mathbf{A} + \mathbf{B}]\boldsymbol{\theta} = \mathbf{0}}$$

$$\det(\lambda \mathbf{A} + \mathbf{B}) = 0$$

State-Space Eigenvalue Problem

The coefficient matrices are real, the *2N* eigenvalues must either be:

- Real
  - very high damping leading to overdamped modes
- Occur in complex conjugate pairs,
  - most structures will have *N* complex conjugate pairs of eigenvalues and corresponding complex conjugate eigenvectors.

Orthogonality Equations for Complex Modes

$$[\lambda_i \mathbf{A} + \mathbf{B}]\boldsymbol{\theta}_i = \mathbf{0}$$

$$\boldsymbol{\theta}_j^{\mathrm{T}}[\lambda_i \mathbf{A} + \mathbf{B}]\boldsymbol{\theta}_i = 0$$

$$[\lambda_j \mathbf{A} + \mathbf{B}]\boldsymbol{\theta}_j = \mathbf{0}$$

$$\boldsymbol{\theta}_i^{\mathrm{T}}[\lambda_j \mathbf{A} + \mathbf{B}]\boldsymbol{\theta}_j = 0$$

$$(\lambda_j - \lambda_i)\,\boldsymbol{\theta}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{\theta}_j = 0$$

$$\boldsymbol{\theta}_i^{\mathrm{T}}\mathbf{A}\boldsymbol{\theta}_j = 0, \qquad \lambda_j \neq \lambda_i, \quad i, j = 1, 2, \ldots, 2N$$

$$\boldsymbol{\theta}_j^{\mathrm{T}}\mathbf{B}\boldsymbol{\theta}_i = 0,$$

Orthogonality Equations for Complex Modes

$$\boxed{\Theta = [\theta_1 \quad \theta_2 \quad \cdots \quad \theta_{2N}]}$$

$$\Theta^T \mathbf{A} \Theta = \mathrm{diag}(a_r), \qquad \Theta^T \mathbf{B} \Theta = \mathrm{diag}(b_r)$$

$$\lambda_r = -\frac{b_r}{a_r}$$

Interpretation of State-Space Eigenvalues

$$\lambda_r \equiv \alpha_r + i\beta_r =$$

$$-\zeta_r\omega_r + i\omega_r\sqrt{1-\zeta_r^2} = -\zeta_r\omega_r + i\omega_{dr}$$

$$\boxed{\omega_r = \sqrt{\alpha_r^2 + \beta_r^2}, \qquad \zeta_r = \frac{-\alpha_r}{\omega_r}}$$

Natural Frequency

Damping Factor

Interpretation of State-Space Eigenvectors: Scaling and Rotating of Complex Eigenvectors

The eigensolution can be expressed in terms of real and imaginary parts by:

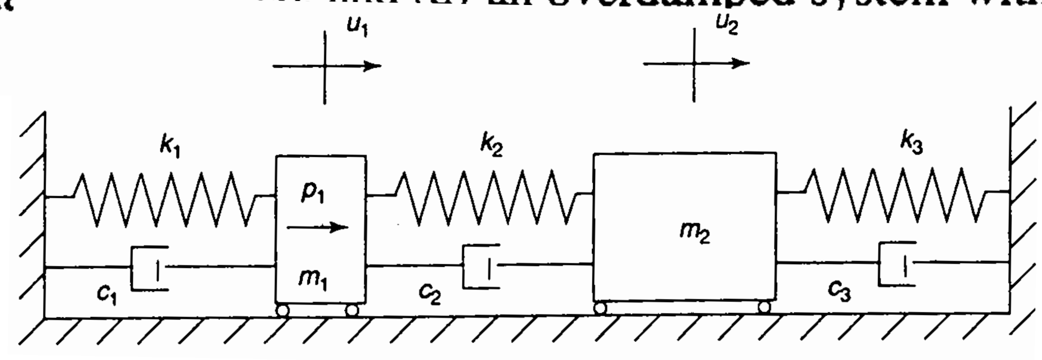
$$\theta e^{\lambda t} = \left\{ \begin{array}{c} \theta_u \\ \lambda \theta_u \end{array} \right\} e^{\lambda t}$$

$$\theta_u = \mathbf{x} + i\mathbf{y}, \qquad \lambda = \alpha + i\beta$$

The following example gives a physical feeling for the results of state-vector eigensolutions.

Interpretation of State-Space Eigenvectors: Scaling and Rotating of Complex Eigenvectors

Example 10.6 For the four listed versions of the 2-DOF spring–mass–dashpot system in Fig. 1, use state-space eigensolutions to determine the natural frequencies, damping factors, and mode shapes. Discuss your solutions. The four systems are: (a) the undamped system, (b) a system with dashpots that produces $\mathbf{C} = 0.5\mathbf{M}$, (c) an underdamped system with a single local damper $c_2 = 20$, and (d) an overdamped system with a single local damper $c_3 = 200$.



$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\mathbf{C} \equiv \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} = \begin{bmatrix} 2200 & -600 \\ -600 & 3800 \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix}$$

(a) Undamped system

$$\lambda = \begin{Bmatrix} 0 + 50.0\,i \\ 0 - 50.0\,i \\ 0 + 40.0\,i \\ 0 - 40.0\,i \end{Bmatrix}$$

$$\Theta = \left[\begin{array}{r|r|r|r} 0 + 0.020\,i & 0 - 0.020\,i & 0 + 0.025\,i & 0 - 0.025\,i \\ 0 - 0.010\,i & 0 + 0.010\,i & 0 + 0.025\,i & 0 - 0.025\,i \\ \hline -1.00 & -1.00 & -1.00 & -1.00 \\ 0.50 & 0.50 & -1.00 & -1.00 \end{array}\right]$$

Observations:

- The eigenvalues are pure imaginary; there is no damping.
- Elements of eigenvector are in phase or out of phase with each other.
- The combination of the two complex-conjugate eigenvectors leads to the real motion of the symmetric mode.

(a) Undamped system

Let us add the top parts of the third and fourth eigenvectors as follows:

$$\theta_{u3}e^{\lambda_3 t} + \theta_{u4}e^{\lambda_4 t} = \begin{Bmatrix} 0.025\,i \\ 0.025\,i \end{Bmatrix} e^{(40.0\mathrm{i})t} + \begin{Bmatrix} -0.025\,i \\ -0.025\,i \end{Bmatrix} e^{(-40.0\mathrm{i})t}$$

$$= \begin{Bmatrix} 0.025\,i \\ 0.025\,i \end{Bmatrix} [\cos(40.0\,t) + i\sin(40.0\,t)]$$

$$+ \begin{Bmatrix} -0.025\,i \\ -0.025\,i \end{Bmatrix} [\cos(40.0\,t) - i\sin(40.0\,t)]$$

$$= -\begin{Bmatrix} 0.050 \\ 0.050 \end{Bmatrix} \sin(40.0\,t)$$

This illustrates why complex eigenvectors must occur in complex-conjugate pairs.

(b) *System with M-proportional damping*

$$\lambda = \begin{Bmatrix} -0.2500 + 49.9994\,i \\ -0.2500 - 49.9994\,i \\ -0.2500 + 39.9992\,i \\ -0.2500 - 39.9992\,i \end{Bmatrix} \quad \omega = \begin{Bmatrix} 50.0 \\ 50.0 \\ 40.0 \\ 40.0 \end{Bmatrix} \quad \zeta = \begin{Bmatrix} 0.0050 \\ 0.0050 \\ 0.0063 \\ 0.0063 \end{Bmatrix}$$

$$[\theta_1 \mid \theta_2] = \left[\begin{array}{cc|cc} 0.0197 & \angle 88.8190 & 0.0197 & \angle -88.8190 \\ 0.0098 & \angle -91.1810 & 0.0098 & \angle 91.1810 \\ \hline 0.9847 & \angle 179.1054 & 0.9847 & \angle -179.1054 \\ 0.4924 & \angle -0.8946 & 0.4924 & \angle 0.8946 \end{array}\right]$$

$$[\theta_3 \mid \theta_4] = \left[\begin{array}{cc|cc} 0.0245 & \angle -91.4762 & 0.0245 & \angle 91.4762 \\ 0.0245 & \angle -91.4762 & 0.0245 & \angle 91.4762 \\ \hline 0.9810 & \angle -1.1181 & 0.9810 & \angle 1.1181 \\ 0.9810 & \angle -1.1181 & 0.9810 & \angle 1.1181 \end{array}\right]$$

$$(88.8190 + 91.1810 = 180.0000)$$

(c) *Local damping with* $C_3 = 20$.

$$\lambda = \begin{Bmatrix} -1.4958 + 49.3641\,i \\ -1.4958 - 49.3641\,i \\ -3.5042 + 40.3448\,i \\ -3.5042 - 40.3448\,i \end{Bmatrix} \quad \omega = \begin{Bmatrix} 49.3868 \\ 49.3868 \\ 40.4967 \\ 40.4967 \end{Bmatrix} \quad \zeta = \begin{Bmatrix} 0.0303 \\ 0.0303 \\ 0.0865 \\ 0.0865 \end{Bmatrix}$$

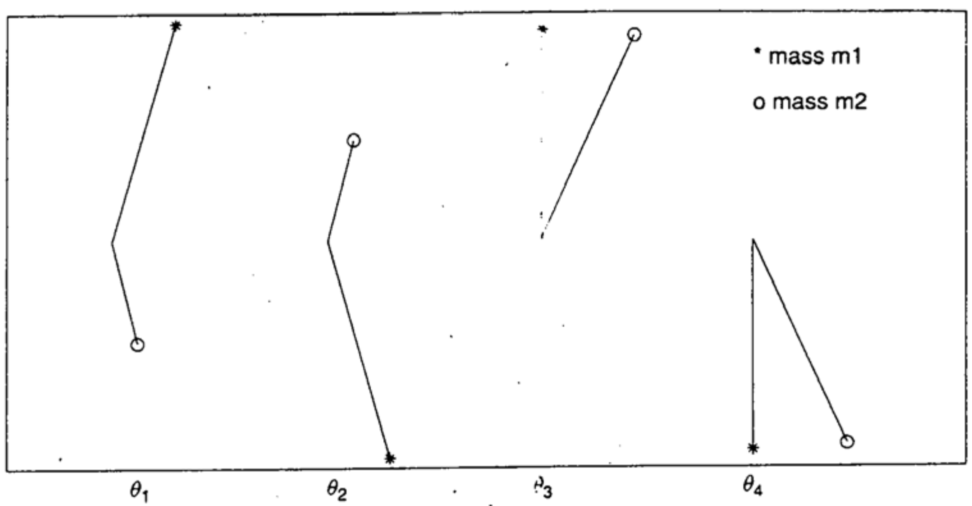
$$[\boldsymbol{\theta}_1 \mid \boldsymbol{\theta}_2] = \left[\begin{array}{c|c} 0.0165 \;\angle 72.9389 & 0.0165 \;\angle -72.9389 \\ 0.0076 \;\angle -74.8685 & 0.0076 \;\angle 74.8685 \\ \hline 0.8138 \;\angle 164.6745 & 0.8138 \;\angle -164.6745 \\ 0.3760 \;\angle 16.8671 & 0.3760 \;\angle -16.8671 \end{array}\right]$$

$$[\boldsymbol{\theta}_3 \mid \boldsymbol{\theta}_4] = \left[\begin{array}{c|c} 0.0177 \;\angle 89.8007 & 0.0177 \;\angle -89.8007 \\ 0.0191 \;\angle 63.9883 & 0.0191 \;\angle -63.9883 \\ \hline 0.7149 \;\angle -175.2353 & 0.7149 \;\angle 175.2353 \\ 0.7737 \;\angle 158.9524 & 0.7737 \;\angle -158.9524 \end{array}\right]$$

(c) *Local damping with* $C_3 = 20$.

Observations:

- In this underdamped system with nonproportional damping, the eigenvalues and eigenvectors occur in two complex-conjugate pairs.
- The natural frequencies are not exactly the same as the natural frequencies of the undamped system
  - However, since this is a fairly lightly damped system, the natural frequencies are close to the respective undamped natural frequencies.
- It has *complex mode shapes.*



(d) *Overdamped system with local damping with* C3 = 200.

$$\lambda = \begin{Bmatrix} -0.4097 + 46.9362\,i \\ -0.4097 - 46.9362\,i \\ -74.9604 \\ -24.2203 \end{Bmatrix} \quad \omega = \begin{Bmatrix} 46.9380 \\ 46.9380 \\ -- \\ -- \end{Bmatrix}, \quad \zeta = \begin{Bmatrix} 0.0087 \\ 0.0087 \\ -- \\ -- \end{Bmatrix}$$

$$[\theta_1 \mid \theta_2] = \left[\begin{array}{ll|ll} 0.0209 & \angle -89.4497 & 0.0209 & \angle 89.4497 \\ 0.0013 & \angle 176.3264 & 0.0013 & \angle -176.3264 \\ \hline 0.9822 & \angle 1.0504 & 0.9822 & \angle -1.0504 \\ 0.0631 & \angle -93.1735 & 0.0631 & \angle 93.1735 \end{array}\right]$$

$$[\theta_3 \mid \theta_4] = \left[\begin{array}{ll|ll} 0.0010 & \angle 180.0000 & 0.0089 & \angle 0 \\ 0.0133 & \angle 180.0000 & 0.0413 & \angle 0 \\ \hline 0.0767 & \angle 0 & 0.2153 & \angle 180.0000 \\ 1.0000 & \angle 0 & 1.0000 & \angle 180.0000 \end{array}\right]$$

(d) *Overdamped system with local damping with* C3 = 200.

Observations:

- There are two large negative real eigenvalues; the system is overdamped to the extent that there is only one pair of complex conjugate eigenvalues and associated eigenvectors.
- The natural frequencies are not exactly the same as natural frequencies
- of the undamped system.
- As can be seen from the eigenvectors (displacement part), the masses are not in phase or 180^0 out of phase. And since this system is heavily damped, it has "very“ complex modes.

Structural Dynamics

Lecture Nine: Vibration Properties of MDOF Systems (Chapter 10)

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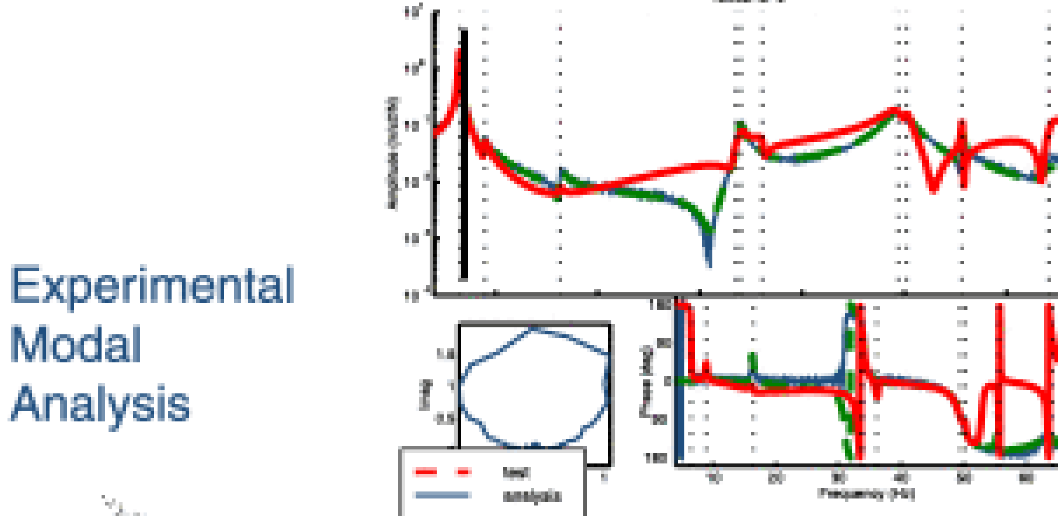

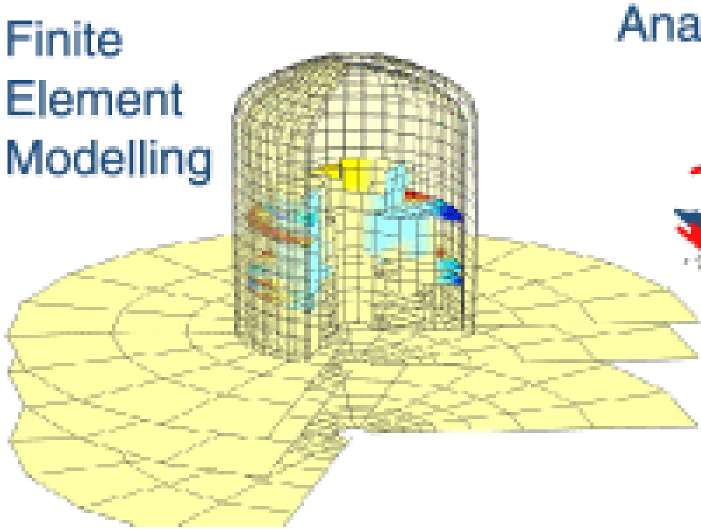

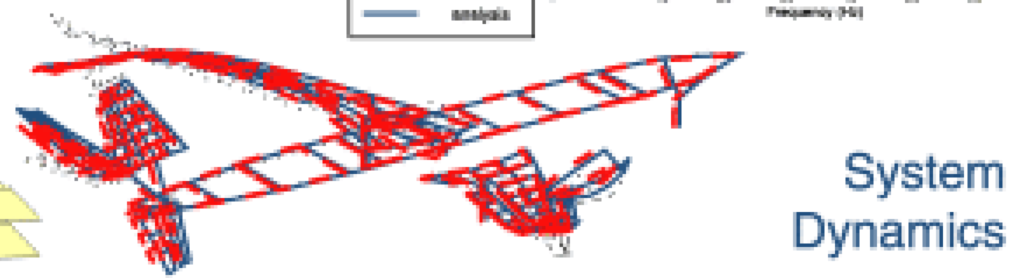

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Vibration Properties of MDOF Systems: Modes, Frequencies, and Damping

1. Some Properties of Natural Frequencies and Natural Modes of Undamped MDOF Systems
2. Model Reduction: Rayleigh, Rayleigh-Ritz, and Assumed-Modes Methods
3. Uncoupled Damping in MDOF Systems
4. Structures with Arbitrary Viscous Damping: Complex Modes
5. Natural Frequencies and Mode Shapes of Damped Structures with Rigid-Body Modes

NATURAL FREQUENCIES AND MODE SHAPES OF DAMPED STRUCTURES WITH RIGID-BODY MODES

- Special treatment is required in determining the state eigenvectors of systems that have rigid-body modes.
- It is necessary to incorporate *generalized eigenvectors* and the *Jordan form* of the eigenvalue matrix.

Generalized Eigenvectors: Jordan Form

- An *NxN* matrix *D* is called *defective* when it fails to have a linearly independent set of *N* eigenvectors (e.g. has a repeated eigenvalues).
  - It is then not possible to transform *D* into diagonal form;

$$\mathbf{D}\boldsymbol{\Phi} = \boldsymbol{\Phi}\boldsymbol{\Lambda}$$

  - Using a linearly independent set of *generalized eigenvectors D is* transformed into block-diagonal *Jordan form*

$$\mathbf{DQ} = \mathbf{QJ}$$

Generalized Eigenvectors: Jordan Form

$$\mathbf{DQ} = \mathbf{QJ} \qquad \mathbf{J} = \left[\begin{array}{c|ccc|cc|c}
\lambda_1 & & & & & & \\
\hline
 & \lambda_2 & 1 & 0 & & & \\
 & 0 & \lambda_2 & 1 & & & \\
 & 0 & 0 & \lambda_2 & & & \\
\hline
 & & & & \lambda_3 & 1 & \\
 & & & & 0 & \lambda_3 & \\
\hline
 & & & & & & \lambda_4
\end{array}\right]$$

The repeated eigenvalues lead to Jordan blocks having the eigenvalue on the diagonal and ones on the superdiagonal.

Undamped Systems with Rigid-Body Modes

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}$$

Θ_{ur} is an $N \times N_r$

$$\mathbf{K}\Theta_{ur} = \mathbf{0}_{nr}$$

$$\mathbf{A}\dot{\mathbf{z}}(t) + \mathbf{B}\mathbf{z}(t) = \mathbf{0} \qquad [\lambda\mathbf{A} + \mathbf{B}]\theta = \mathbf{0}$$

$$\mathbf{B}\theta \equiv \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{Bmatrix} \theta_u \\ \theta_v \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \quad \theta'_r = \begin{Bmatrix} \theta_u \\ \mathbf{0} \end{Bmatrix}$$

Undamped Systems with Rigid-Body Modes

- There are only N_r rigid body modes.
- Zero eigenvalues must occur with multiplicity $2N_r$
- The generalized eigenproblem is defective.
- For each regular state rigid-body mode, there will be a corresponding *generalized state rigid-body mode*

$$\mathbf{A}[\theta_r', \theta_r''] \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \mathbf{B}[\theta_r', \theta_r''] = [0\ 0]$$

Generalized State Rigid-Body Mode

$$\mathbf{A}[\theta'_r\ \ \theta''_r]\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \mathbf{B}[\theta'_r\ \ \theta''_r] = [0\ \ 0]$$

regular state rigid-body modes *generalized state rigid-body mode*

$$\mathbf{B}\Theta'_r = \mathbf{0}_{sr}$$

$$\mathbf{B}\Theta''_r = -\mathbf{A}\Theta'_r$$

Generalized State Rigid-Body Mode

$$\begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \Theta''_{ur} \\ \Theta''_{vr} \end{bmatrix} = -\begin{bmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Theta_{ur} \\ \mathbf{0}_{nr} \end{bmatrix}$$

$$\mathbf{K}\Theta''_{ur} = \mathbf{0}_{nr}$$

$$-\mathbf{M}[\Theta''_{vr} - \Theta_{ur}] = \mathbf{0}_{nr} \longrightarrow \Theta''_{vr} = \Theta_{ur}$$

The regular state rigid-body mode contains the displacement representation, it is sufficient to set

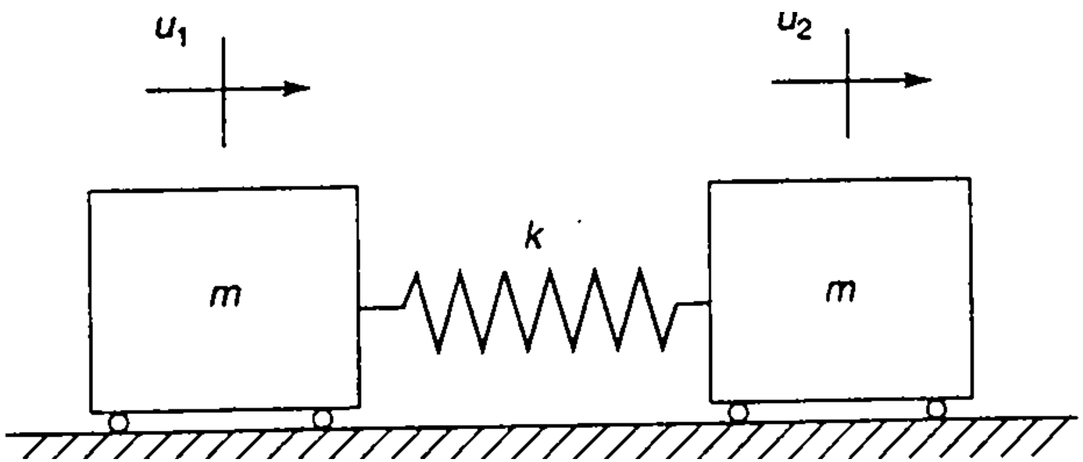
$$\Theta''_{ur} = \mathbf{0}_{nr}.$$

Generalized State Rigid-Body Mode

The complete set of *state rigid-body modes* for an undamped system is given by:

$$\boxed{\boldsymbol{\Theta}_r \equiv \left[\boldsymbol{\Theta}'_r \;\; \boldsymbol{\Theta}''_r\right] = \begin{bmatrix} \boldsymbol{\Theta}_{ur} & \mathbf{0}_{nr} \\ \mathbf{0}_{nr} & \boldsymbol{\Theta}_{ur} \end{bmatrix}}$$

Example



$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\lambda = 0,\ 0,\ i\sqrt{2},\ -i\sqrt{2}$$

Example

The rank of $[\lambda_1\mathbf{A} + \mathbf{B}]$ is 3, there will only be one regular state eigenvector given by:

$$\mathbf{B}\boldsymbol{\theta}_1 = \mathbf{0} \qquad \boldsymbol{\theta}_1^T = [1\ 1\ 0\ 0]$$

$$\mathbf{B}\boldsymbol{\theta}_2 = -\mathbf{A}\boldsymbol{\theta}_1 \qquad \boldsymbol{\theta}_2^T = [0\ 0\ 1\ 1]$$

$$\boldsymbol{\Theta} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ 0 & 1 & i\sqrt{2} & -i\sqrt{2} \\ 0 & 1 & -i\sqrt{2} & i\sqrt{2} \end{bmatrix} \qquad \mathbf{J} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & i\sqrt{2} & 0 \\ 0 & 0 & 0 & -i\sqrt{2} \end{array}\right]$$

Viscous-Damped Systems with Rigid-Body Modes

$$[\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}]\,\theta_u = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \Theta''_{ur} \\ \Theta''_{vr} \end{bmatrix} = -\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Theta_{ur} \\ \mathbf{0}_{nr} \end{bmatrix}$$

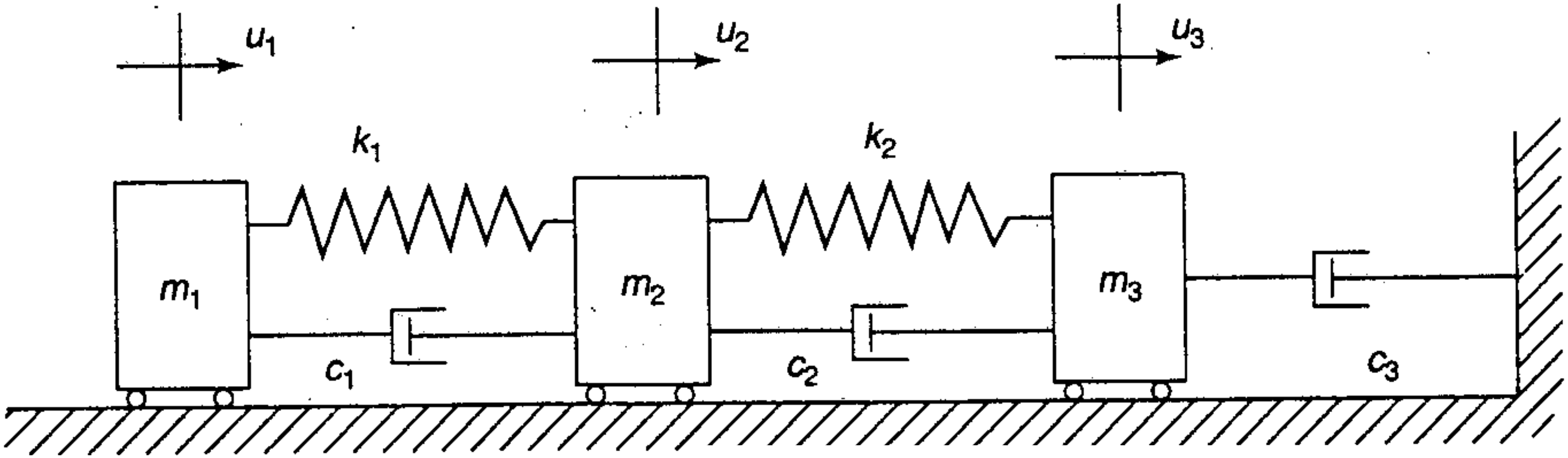
$$\mathbf{K}\Theta''_{ur} = -\mathbf{C}\,\Theta_{ur} \longrightarrow \text{Depends on } C$$

$$-\mathbf{M}[\Theta''_{vr} - \Theta_{ur}] = \mathbf{0}_{nr} \longrightarrow \Theta''_{vr} = \Theta_{ur}$$

Viscous-Damped Systems with Rigid-Body Modes

- If any of the columns of $\mathbf{C}\dot{\Theta}_{ur}$, are zero, there will be a solution for the corresponding column of $\mathbf{K}\Theta''_{ur}$ just as for the undamped case.
- For those columns of $\mathbf{C}\dot{\Theta}_{ur}$ that are not zero, there will not be a solution for the corresponding columns of $\mathbf{K}\Theta''_{ur}$
- Hence, the assumption that $\lambda = 0$ is a double root leading to both regular and generalized state rigid-body modes is not valid.

Example



$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Determine the eigenvectors corresponding to zero eigenvalues.

(a) *Case for which* $\mathbf{C}\boldsymbol{\theta}_{ur} = \mathbf{0}$.

$$\mathbf{C} = \begin{bmatrix} 0.2 & -0.2 & 0.0 \\ -0.2 & 0.6 & -0.4 \\ 0.0 & -0.4 & 0.4 \end{bmatrix}$$

$$\lambda^2(\lambda^4 + 1.2\lambda^3 + 4.24\lambda^2 + 1.8\lambda + 3) = 0$$

$$[\boldsymbol{\theta}_1 \ \boldsymbol{\theta}_2] = \left[\begin{array}{cc} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ \hline 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{array}\right]$$

(b) *Case for which* $\mathbf{C}\boldsymbol{\theta}_{ur} \neq \mathbf{0}$.

$$\mathbf{C} = \begin{bmatrix} 0.2 & -0.2 & 0.0 \\ -0.2 & 0.6 & -0.4 \\ 0.0 & -0.4 & 0.6 \end{bmatrix}$$

$$\dot{\lambda}\,(\lambda^5 + 1.4\lambda^4 + 4.4\lambda^3 + 2.416\lambda^2 + 3.12\lambda + 0.2) = 0$$

- The rigid-body displacement mode is the same as for previous case

$$\boldsymbol{\theta}_{u1} = [\,1 \;\; 1 \;\; 1\,]^T$$

- No generalized eigenvector corresponding to $\lambda = 0$

Structural Dynamics

Lecture Ten: Dynamic Response of MDOF Systems: (Chapter 11)

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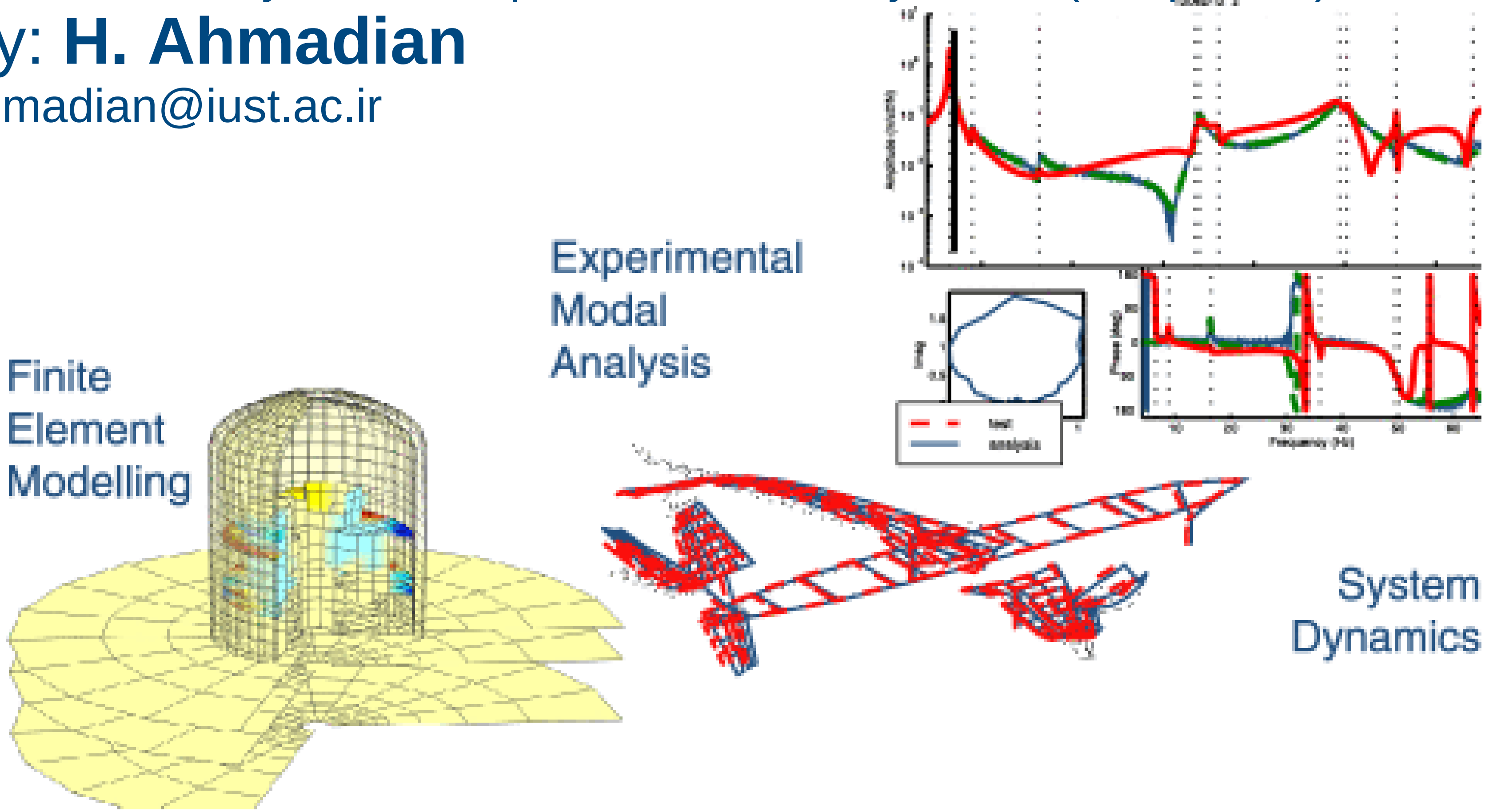

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Dynamic Response of MDOF Systems: Mode-Superposition Method

- Mode-Superposition Method: Principal Coordinates
- Mode-Superposition Solutions for MDOF Systems with Modal Damping: Frequency-Response Analysis
- Mode-Displacement Solution for the Response of MDOF Systems
- Mode-Acceleration Solution for the Response of Undamped MDOF Systems
- Dynamic Stresses by Mode Superposition
- Mode Superposition for Undamped Systems with Rigid-Body Modes

MODE-SUPERPOSITION METHOD: PRINCIPAL COORDINATES

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{p}(t)$$

$$\boxed{\mathbf{u}(t) = \mathbf{\Phi}\boldsymbol{\eta}(t) = \sum_{r=1}^{N} \boldsymbol{\phi}_r \eta_r(t)}$$

$$\boldsymbol{M}\ddot{\boldsymbol{\eta}} + \boldsymbol{C}\dot{\boldsymbol{\eta}} + \boldsymbol{K}\boldsymbol{\eta} = \boldsymbol{f}(t)$$

$$\boxed{\begin{aligned} \boldsymbol{M} &= \mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi} = \text{modal mass matrix} = \operatorname{diag}(M_r) \\ \boldsymbol{C} &= \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi} = \text{modal damping matrix} \\ \boldsymbol{K} &= \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \text{modal stiffness matrix} = \operatorname{diag}(\omega_r^2 M_r) \\ \boldsymbol{f}(t) &= \mathbf{\Phi}^T\mathbf{p}(t) = \text{modal force vector} \end{aligned}}$$

Modal Damping: Uncoupled Equations of Motion

$$C = \Phi^T \mathbf{C} \Phi = \text{diag}(C_r) = \text{diag}(2\zeta_r \omega_r M_r)$$

$$\boxed{M_r \ddot{\eta}_r + 2M_r \omega_r \zeta_r \dot{\eta}_r + \omega_r^2 M_r \eta_r = f_r(t), \qquad r = 1, 2, \ldots, N}$$

Initial Conditions in Modal Coordinates

$$\mathbf{u}(0) = \Phi \eta(0), \qquad \dot{\mathbf{u}}(0) = \Phi \dot{\eta}(0)$$

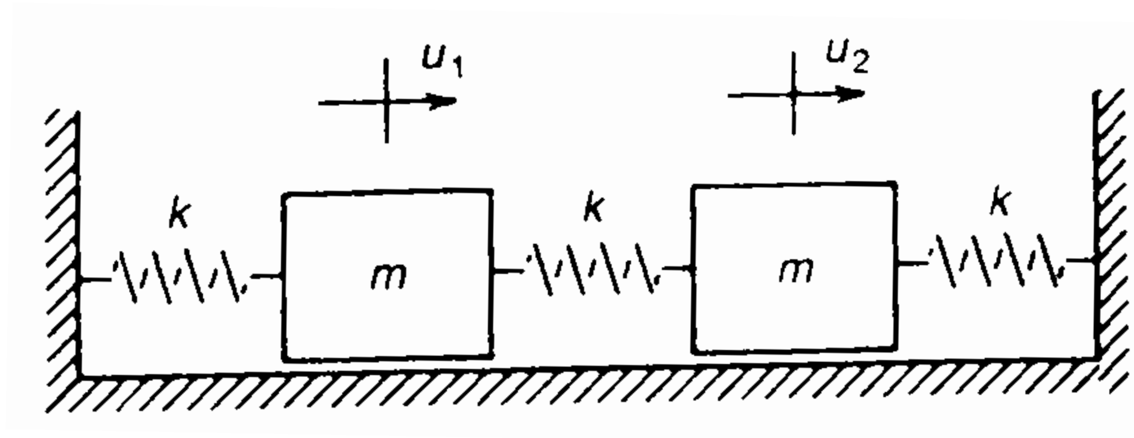
$$\Phi^T \mathbf{M} \mathbf{u}(0) = M \eta(0), \qquad \Phi^T \mathbf{M} \dot{\mathbf{u}}(0) = M \dot{\eta}(0)$$

Mode-Superposition Solution for Free Vibration of an Undamped MDOF System

$$\boxed{\mathbf{u}(t) = \sum_{r=1}^{N} \boldsymbol{\phi}_r (a_r \cos \omega_r t + b_r \sin \omega_r t)}$$

$$\boxed{a_r = \frac{\boldsymbol{\phi}_r^{\mathrm{T}} \mathbf{M} \mathbf{u}(0)}{M_r}, \qquad b_r = \frac{\boldsymbol{\phi}_r^{\mathrm{T}} \mathbf{M} \dot{\mathbf{u}}(0)}{M_r \omega_r}}$$

Example: The free vibration response of a 2-DOF system



$$\mathbf{u}(0) = \begin{Bmatrix} 0 \\ u_0 \end{Bmatrix}, \qquad \dot{\mathbf{u}}(0) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \qquad \boldsymbol{\phi}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \qquad \boldsymbol{\phi}_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}.$$

Example: The free vibration response ...

$$M_1 = [\,1 \quad 1\,]\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 2m \qquad \mathbf{Mu}(0) = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} 0 \\ u_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ mu_0 \end{Bmatrix}$$

$$M_2 = [\,1 \quad -1\,]\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = 2m \qquad \mathbf{M\dot{u}}(0) = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$a_1 = \frac{\boldsymbol{\phi}_1^{\mathrm{T}}\mathbf{Mu}(0)}{M_1} = \frac{1}{2m}[\,1 \quad 1\,]\begin{Bmatrix} 0 \\ mu_0 \end{Bmatrix} = \frac{u_0}{2}$$

$$a_2 = \frac{\boldsymbol{\phi}_2^{\mathrm{T}}\mathbf{Mu}(0)}{M_2} = \frac{1}{2m}[\,1 \quad -1\,]\begin{Bmatrix} 0 \\ mu_0 \end{Bmatrix} = -\frac{u_0}{2}$$

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \frac{u_0}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\cos\omega_1 t - \frac{u_0}{2}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}\cos\omega_2 t$$

MODE-SUPERPOSITION SOLUTIONS FOR MDOF SYSTEMS WITH MODAL DAMPING: FREQUENCY-RESPONSE ANALYSIS

$$\ddot{\eta}_r + 2\zeta_r\omega_r\dot{\eta}_r + \omega_r^2\eta_r = \frac{1}{M_r}f_r(t) = \frac{1}{M_r}\boldsymbol{\phi}_r^T\mathbf{p}(t)$$

$$\mathbf{p}(t) = \mathbf{P}\cos\Omega t \qquad F_r = \boldsymbol{\phi}_r^T\mathbf{P}$$

$$\ddot{\overline{\eta}}_r + 2\zeta_r\omega_r\dot{\overline{\eta}}_r + \omega_r^2\overline{\eta}_r = \omega_r^2\frac{F_r}{K_r}e^{i\Omega t}$$

$$\boxed{\eta_r(t) = \frac{F_r/K_r}{\sqrt{(1-r_r^2)^2 + (2\zeta_r r_r)^2}}\cos(\Omega t - \alpha_r)} \qquad \boxed{\tan\alpha_r = \frac{2\zeta_r r_r}{1-r_r^2}}$$

MODE-SUPERPOSITION SOLUTIONS FOR MDOF SYSTEMS WITH MODAL DAMPING: FREQUENCY-RESPONSE ANALYSIS

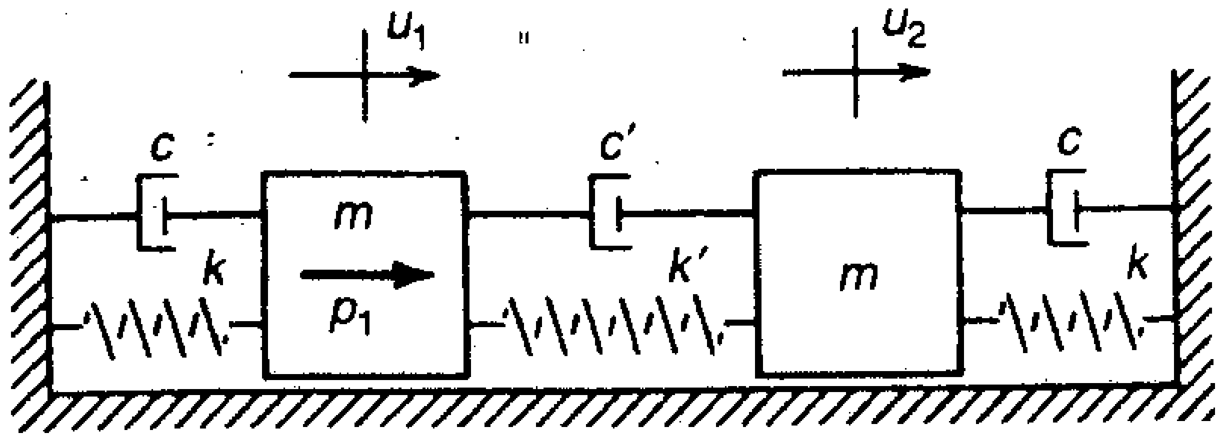
$$\bar{\mathbf{u}}(t) = \mathbf{\Phi}\bar{\eta}(t) = \sum_{r=1}^{N} \boldsymbol{\phi}_r \bar{\eta}_r(t)$$

$$\bar{\mathbf{u}}(t) = \sum_{r=1}^{N} \frac{\boldsymbol{\phi}_r \boldsymbol{\phi}_r^{\mathrm{T}} \mathbf{P}}{K_r} \frac{1}{(1 - r_r^2) + i(2\zeta_r r_r)} e^{i\Omega t}$$

The *complex frequency-response function* (FRF)

$$\boxed{\overline{H}_{ij}(\Omega) \equiv \overline{H}_{u_i/p_j}(\Omega) = \sum_{r=1}^{N} \frac{\phi_{ir}\phi_{jr}}{K_r} \frac{1}{(1 - r_r^2) + i(2\zeta_r r_r)}}$$

Example:



$$p_1(t) = P_1 \cos \Omega t, \; k = 987,$$

$$k' = 217, \; m = 1, \; c = 0.6284, \; \text{and} \; c' = 0.0628$$

- Determine the system's modes
- Determine the modal mass/stiffness/damping
- Determine the complex frequency-response functions

Example: the system's modes

$$\left[\begin{bmatrix} k+k' & -k' \\ -k' & k+k' \end{bmatrix} - \omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\right] \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\omega_1^2 = \frac{k}{m}, \qquad \omega_2^2 = \frac{k+2k'}{m}$$

$$f_1 = \frac{\omega_1}{2\pi} = 5.00 \text{ Hz}, \qquad f_2 = \frac{\omega_2}{2\pi} = 6.00 \text{ Hz}$$

$$\phi_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \qquad \phi_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Example: the modal properties

$$\boldsymbol{M} = \mathbf{\Phi}^{T}\mathbf{M}\mathbf{\Phi} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$K_1 = \omega_1^2 M_1 = 987(2) = 1974,$$
$$K_2 = \omega_2^2 M_2 = 1421(2) = 2842$$

$$\boldsymbol{C} = \mathbf{\Phi}^{T}\mathbf{C}\mathbf{\Phi} = \begin{bmatrix} 1.2568 & 0 \\ 0 & 1.5080 \end{bmatrix}$$

$$\zeta_r = \frac{C_r}{2M_r\omega_r}$$

$$\zeta_1 = \frac{1.2568}{2(2)(31.42)} = 0.0100,$$

$$\zeta_2 = \frac{1.5080}{2(2)(37.70)} = 0.0100$$

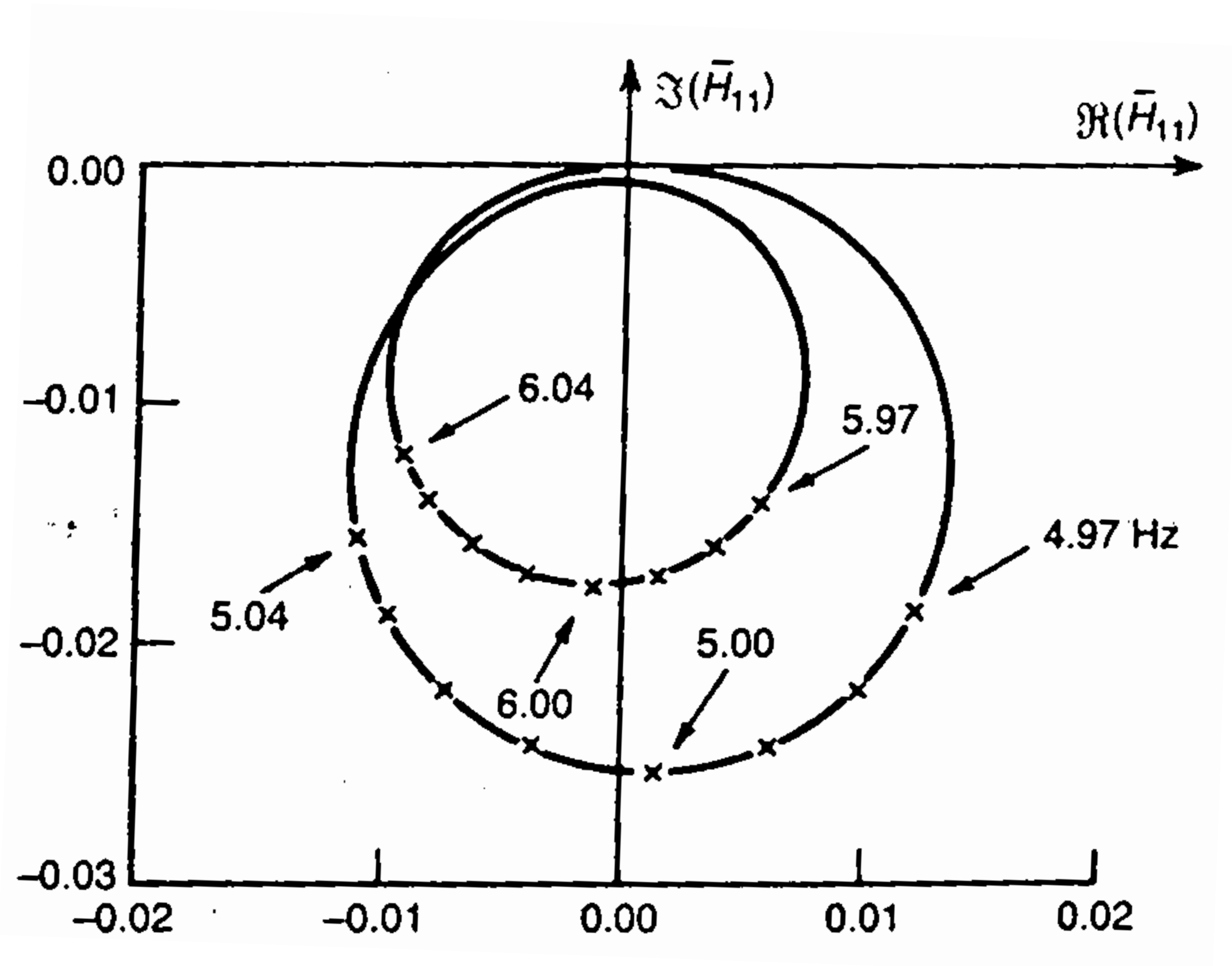
Example: frequency-response functions

$$\overline{H}_{ij}(\Omega) = \sum_{r=1}^{2} \frac{\phi_{ir}\phi_{jr}}{K_r} \frac{1}{1-(\Omega/\omega_r)^2 + i(2\zeta_r\Omega/\omega_r)}$$

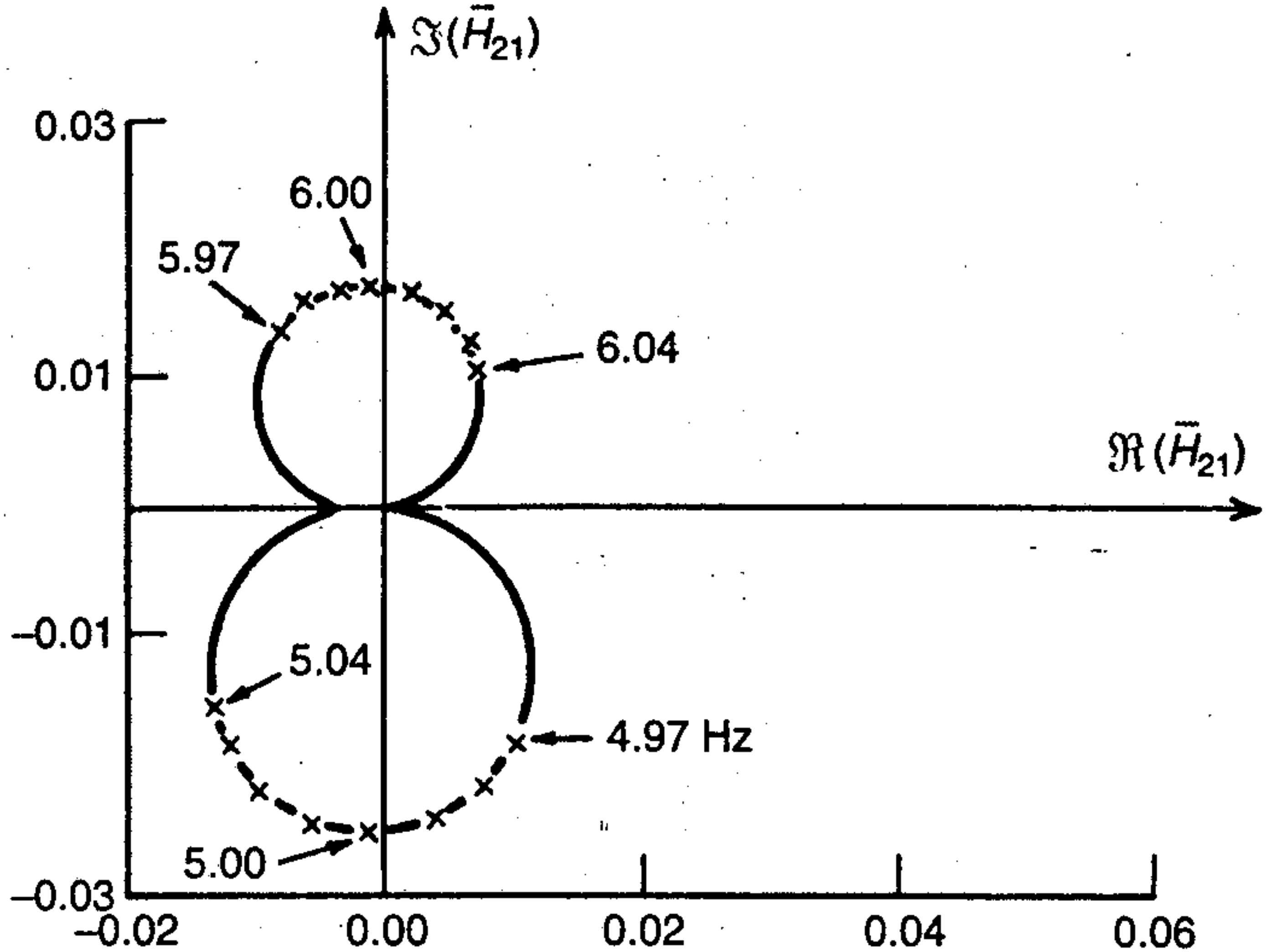
$$\overline{H}_{11}(\Omega) = \frac{5.066\times 10^{-4}}{1-(\Omega/31.42)^2 + i\,(0.02\Omega/31.42)} + \frac{3.519\times 10^{-4}}{1-(\Omega/37.70)^2 + i\,(0.02\Omega/37.70)}$$

$$\overline{H}_{21}(\Omega) = \frac{5.066\times 10^{-4}}{1-(\Omega/31.42)^2 + i\,(0.02\Omega/31.42)} - \frac{3.519\times 10^{-4}}{1-(\Omega/37.70)^2 + i\,(0.02\Omega/37.70)}$$

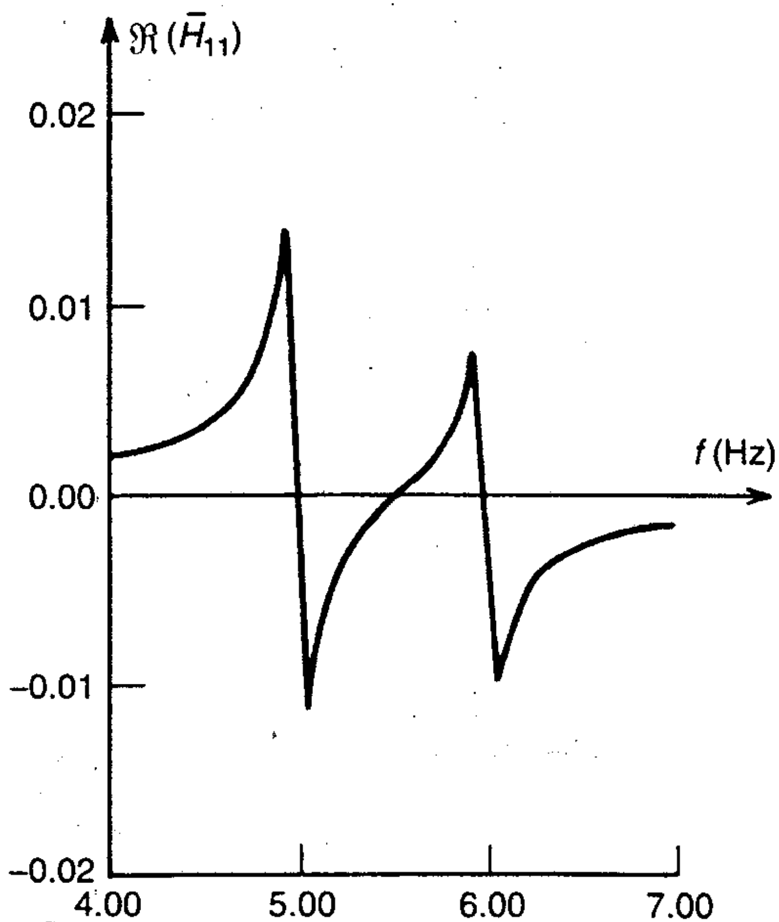
Example: frequency-response functions

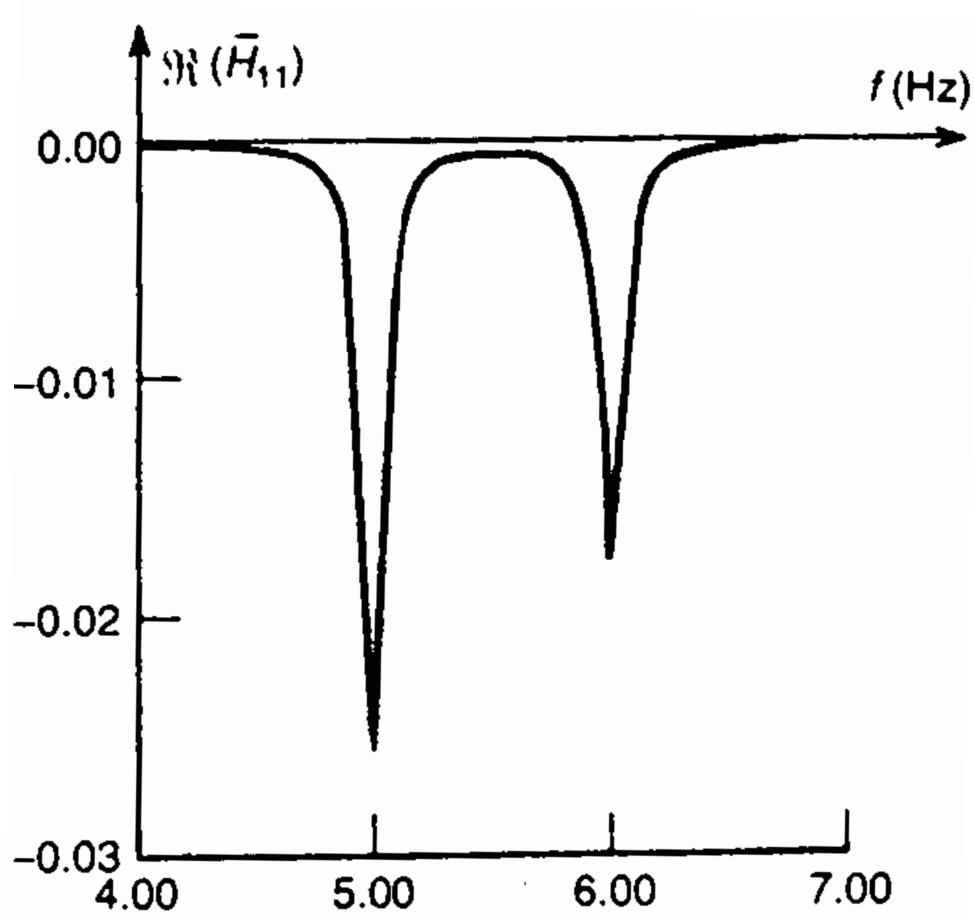


Example: frequency-response functions

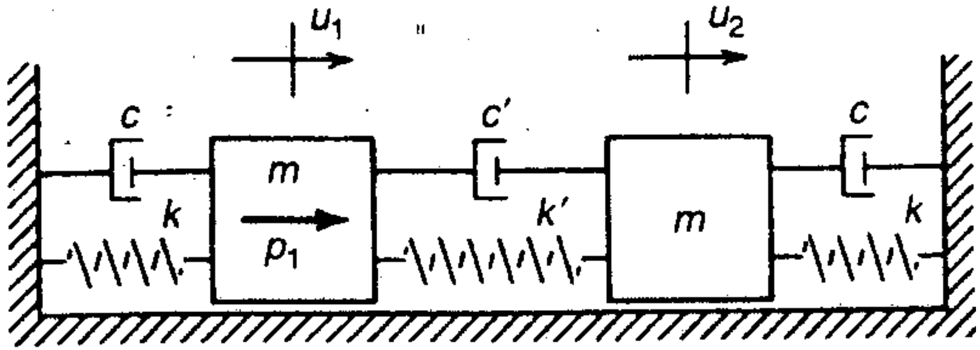


Example: frequency-response functions





Example:



$$k = 987, \qquad k' = 10, \qquad m = 1,$$

$$c = 0.6284, \qquad c' = 0.0031$$

- Repeat the same example with the new set of parameters. *(Note:* These parameters were chosen to give 1% damping and to give natural frequencies separated by only 1%.)

Example: the system's modal properties

$$f_1 = \frac{\omega_1}{2\pi} = 5.00 \text{ Hz}, \qquad f_2 = \frac{\omega_2}{2\pi} = 5.05 \text{ Hz}$$

$$\boldsymbol{\Phi} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{M}\boldsymbol{\Phi} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

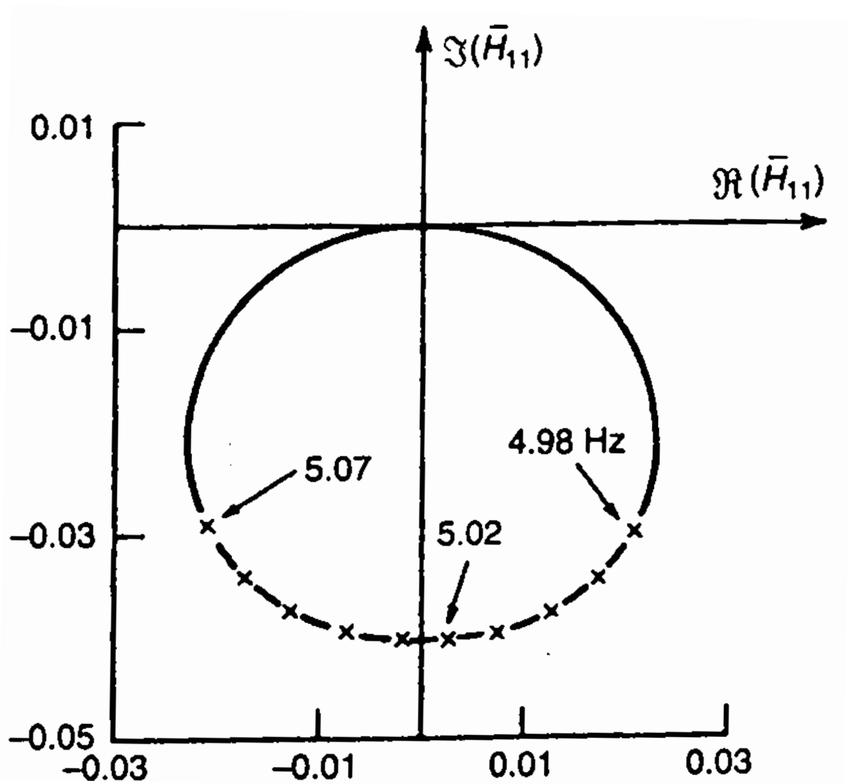
$$\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{C}\boldsymbol{\Phi} = \begin{bmatrix} 1.2568 & 0 \\ 0 & 1.2692 \end{bmatrix}$$

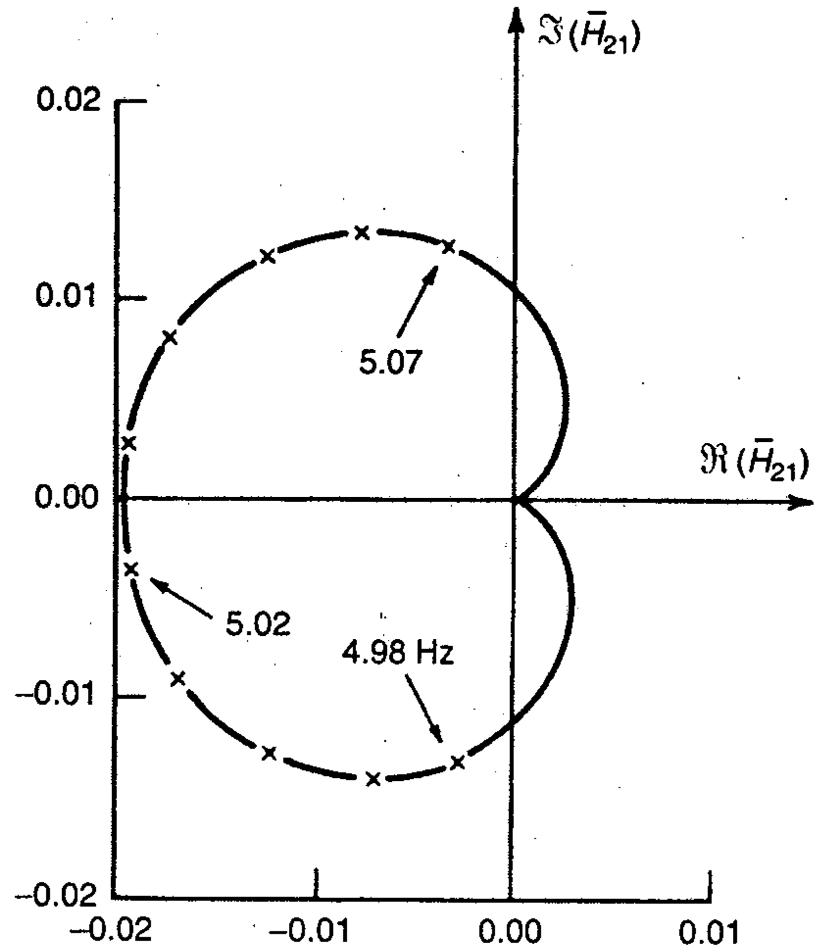
$$K = \begin{bmatrix} 1974 & 0 \\ 0 & 2014 \end{bmatrix}$$

$$\zeta_1 = \frac{1.2568}{2(2)(31.42)} = 0.0100,$$

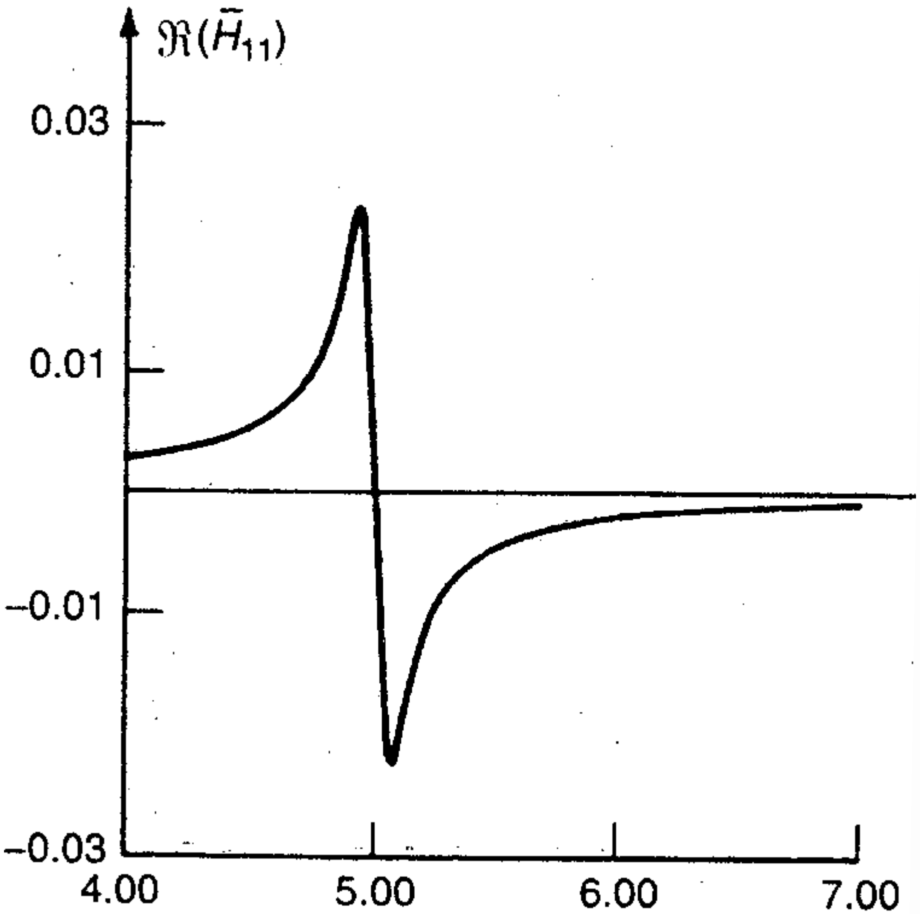
$$\zeta_2 = \frac{1.2692}{2(2)(31.73)} = 0.0100$$

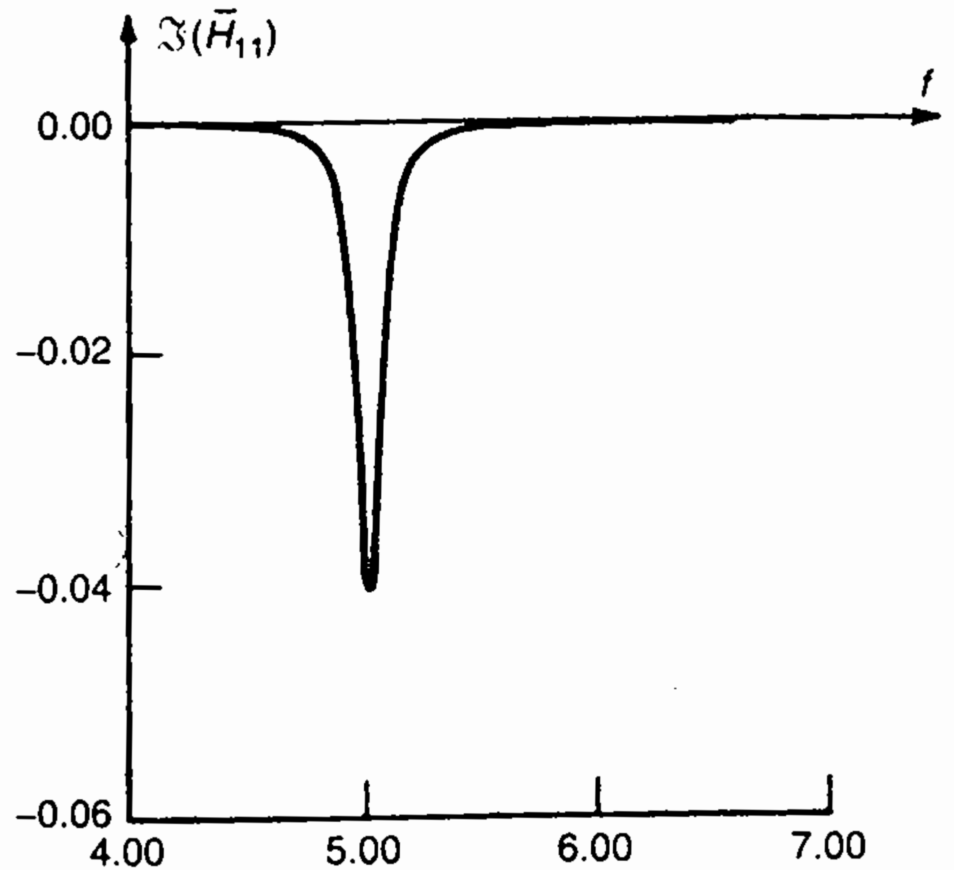
Example: frequency-response functions





Example: frequency-response functions





Dynamic Response of MDOF Systems: Mode-Superposition Method

- Mode-Superposition Method: Principal Coordinates
- Mode-Superposition Solutions for MDOF Systems with Modal Damping: Frequency-Response Analysis
- Mode-Displacement Solution for the Response of MDOF Systems
- Mode-Acceleration Solution for the Response of Undamped MDOF Systems
- Dynamic Stresses by Mode Superposition
- Mode Superposition for Undamped Systems with Rigid-Body Modes