Chapter 6:
Deflections and Stability

6 DEFLECTIONS AND STABILITY

6.1 INTRODUCTION

Methods for finding the collapse loads of steel frames were examined in detail in chapters 3, 4 and to some extent, 5. In the virtual work method, for example, the collapse load was found by considering small (virtual) deformations of the collapse mechanism. However, the shape of the structure before deformation of the mechanism was assumed to be the same as when there was no load on the structure. In other words, all deformation before collapse was ignored. There must be deformation before collapse, but how significant is it?

It is not sufficient to ensure that the structure is strong enough to resist the applied loading with an adequate load factor against collapse. It is also necessary to make certain that deflections do not become excessive. Consider the pitched portal frame shown in figure 6.1. The frame carries the rails for an overhead travelling crane. The crane-wheels which run on the rails will only have a finite amount of sideways movement. Consequently the dimension L is a critical part of the design. If it changes too much, due to deflection of the frame, the wheels will jam and the crane will no longer be travelling.

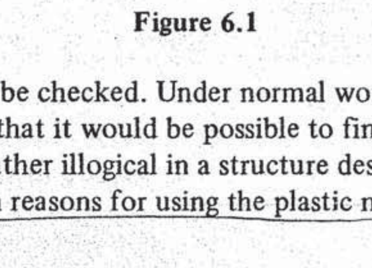


Figure 6.1

The deflections must be checked. Under normal working loads the structure should still be elastic so that it would be possible to find the deflections by elastic analysis. This is rather illogical in a structure designed by plastic methods: after all, one of the main reasons for using the plastic methods is that they avoid the tedious calculations of the elastic methods. There is another reason for being wary of the elastic methods. Figure 6.2 shows a possible load deflection curve for the frame in figure 6.1. Although the structure would be elastic at working loads, plastic design would produce a structure with bending moments close to the plastic moment at certain critical sections. It would only need a small overload on the crane for plastic hinges to form with a large increase in the deflections. Overhead cranes are notorious for overloading — after all, it is quicker to make one lift rather than two! A logical solution would be to limit the actual deflections at the point of collapse, especially if they can be calculated conveniently.

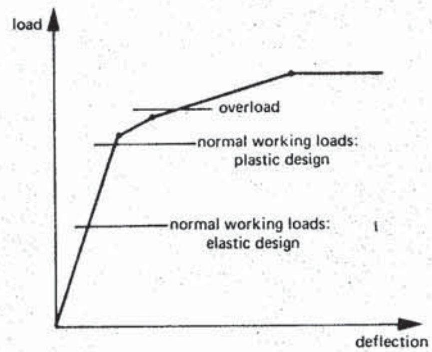


Figure 6.2

There is another possibility. The deflections before collapse may significantly reduce the collapse load of the structure. It is well known that deflections reduce the stiffness of struts (the so-called P–δ effect [14]) and the same thing can happen to frames. The problem is more serious in more flexible structures, where the deflections can cause an unexpected collapse mechanism due to overall buckling (instability) of the frame.

Originally only mild steel structures were designed by plastic methods, but nowadays higher strength steel structures are handled in the same way. The reduced ductility (see figure 1.5) is usually adequate for the formation of a collapse mechanism, but the higher yield stress means that smaller sections than would be required with mild steel can be used. This results in a more flexible structure with larger deflections. Obviously the problems caused by deflections are likely to be more severe. This must always be borne in mind when using higher strength steels.

The first part of this chapter describes a straightforward method for calculating deflections at the point of collapse. The second part is an examination of the effects of deflection on the collapse load.

6.2 CALCULATION OF DEFLECTIONS AT THE POINT OF COLLAPSE

6.2.1 Background Theory

One essential assumption that was used in finding collapse loads was that all plastic rotation occurs at the plastic hinges. This means that between the plastic hinges the members are elastic. Any frame can be broken down, therefore, into individual elastic members, with all plastic behaviour occurring at the ends of the members. (There will be some members, of course, whose end moments are less than the plastic moment.)

The deflections of the structure can be represented by the displacements of the ends of each member. Since the members are elastic, the end moments and the *elastic* end displacements can be related by the slope deflection equations.[1] Using the notation in figure 6.3 these are

$$M_{AB} = \frac{EI}{L}\left(4\theta_{AB} + 2\theta_{BA} - 6\frac{\delta}{L}\right) + (FEM)_{AB}$$
$$M_{BA} = \frac{EI}{L}\left(2\theta_{AB} + 4\theta_{BA} - 6\frac{\delta}{L}\right) + (FEM)_{BA} \quad (6.1)$$

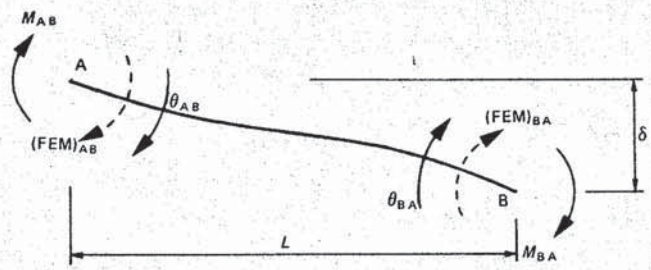


Figure 6.3

The diagram shows the positive sense of moments and deformations. This *clockwise positive convention* has been observed in the following examples. Equation 6.1 can be rearranged to give

$$\theta_{AB} = \frac{\delta}{L} + \frac{L}{6EI}(2M_{AB} - M_{BA}) - \frac{L}{6EI}[2(FEM)_{AB} - (FEM)_{BA}]$$
$$\theta_{BA} = \frac{\delta}{L} + \frac{L}{6EI}(-M_{AB} + 2M_{BA}) - \frac{L}{6EI}[-(FEM)_{AB} + 2(FEM)_{BA}] \quad (6.2)$$

The modified slope deflection equations 6.2 can be used to find deflections at the point of collapse, when the final plastic hinge has just formed but has not started to rotate. It is easiest to explain the process by looking at examples.

6.2.2 Fixed End Beam with UDL

The beam and various stages of the analysis are shown in figure 6.4.

First Stage

The first stage of the analysis is to determine the collapse mechanism and collapse load (or load factor).

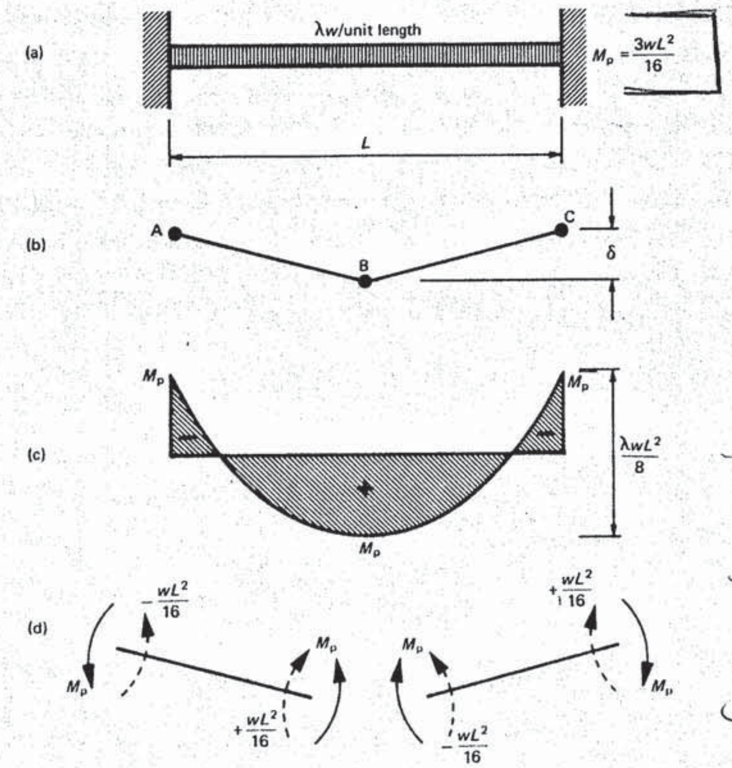


Figure 6.4

In this example, the problem is symmetric and the free and reactant BM method can be used. The mechanism and bending moment diagram are shown in figures 6.4b and c. From the geometry of the BMD

$$\frac{\lambda_c w L^2}{8} = 2M_p$$

so that when $M_p = 3wL^2/16$

$$\lambda_c = 3$$

Second Stage

It is now necessary to divide the structure into individual (elastic) members and write down the modified slope deflection equations for each member.

The beam can be divided into two members, AB and BC, between the plastic hinges. The end moments for both members are equal to the plastic moment M_p. The difficulty is to determine the direction in which they act. This can be done by noting that the end moments resist the end rotations, therefore they must act in *the opposite sense to the plastic rotations.* The fixed end moments in AB and BC are the standard case of a UDL ($3w$ per unit length) on a fixed beam (of span $L/2$). The various moments are summarised in figure 6.4d.

i.e $\frac{qL^2}{12} = \frac{3w(\frac{L}{2})^2}{12} = \frac{wL^2}{16}$

i.e moments oppose the rotations.

The slope deflection equations can now be written down for each member

$$\theta_{AB} = \frac{2\delta}{L} + \frac{L}{12EI}(-2M_p + M_p) - \frac{L}{12EI}\left(-\frac{wL^2}{8} - \frac{wL^2}{16}\right)$$

$$\theta_{AB} = \frac{2\delta}{L} - \frac{M_pL}{12EI} + \frac{wL^3}{64EI}$$

$$\theta_{BA} = \frac{2\delta}{L} + \frac{L}{12EI}(M_p - 2M_p) - \frac{L}{12EI}\left(\frac{wL^2}{16} + \frac{wL^2}{8}\right)$$

$$\theta_{BA} = \frac{2\delta}{L} - \frac{M_pL}{12EI} - \frac{wL^3}{64EI}$$

$$\theta_{BC} = -\frac{2\delta}{L} + \frac{L}{12EI}(2M_p - M_p) - \frac{L}{12EI}\left(-\frac{wL^2}{8} - \frac{wL^2}{16}\right)$$

$$\theta_{BC} = -\frac{2\delta}{L} + \frac{M_pL}{12EI} + \frac{wL^3}{64EI}$$

and similarly

$$\theta_{CB} = -\frac{2\delta}{L} + \frac{M_pL}{12EI} + \frac{wL^3}{64EI}$$

Notice the term $-2\delta/L$ in the last two equations. The sign convention in figure 6.3 defined the deflection δ as positive when the right-hand end (B) sank below the left-hand end (A), causing a clockwise rotation of the whole member. In the example, BC is rotating anticlockwise, hence the negative sign.

Third Stage

The deflection must now be calculated when the last plastic hinge has just formed. But which is the last hinge? There is no way of knowing this, so each hinge in turn must be assumed to form last, and a deflection calculated for each one.

If the hinge at A (or at C because of symmetry) is the last to form, there will have been no rotation at A (or C) because it is a clamped end. Thus

$$\theta_{AB} = \frac{2\delta}{L} - \frac{M_pL}{12EI} + \frac{wL^3}{64EI} = 0$$

so that

$$\frac{2\delta}{L} = \frac{M_pL}{12EI} - \frac{wL^3}{64EI} = \frac{wL^3}{EI}\left(\frac{3}{12.16} - \frac{1}{64}\right)$$

$$\delta = 0$$

If the hinge at B is the last to form, there will have been no plastic rotation at B at the point of collapse. The whole beam (AC) will still be continuous at B. This can only be achieved if

$$\theta_{BA} = \theta_{BC}$$

Substituting for θ_{BA} and θ_{BC} gives

$$\frac{2\delta}{L} - \frac{M_pL}{12EI} - \frac{wL^3}{64EI} = -\frac{2\delta}{L} + \frac{wL^3}{64EI}$$

$$\frac{4\delta}{L} = \frac{2M_pL}{12EI} + \frac{2wL^3}{64EI} = \frac{2wL^3}{EI}\left(\frac{3}{12 \times 16} + \frac{1}{64}\right)$$

$$= \frac{4wL^3}{64EI}$$

$$\delta = \frac{wL^4}{64EI}$$

The question now is which of these values is correct?

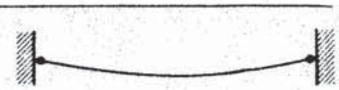
Fourth Stage

One way to decide is to substitute the values of δ back into the slope deflection equations and obtain the rotations.

The results of this are summarised in table 6.1. Inspection of the table shows that the first deflected shape is ridiculous. The only conclusion is that the last hinge forms at B and the deflection at the point of collapse is $wL^4/64EI$. In a more complicated structure it would be tedious to set up a much enlarged version of table 6.1. Instead the deflection can be chosen by using the displacement theorem. This states *'Let displacements be predicted on the basis of each plastic hinge forming last. If in the loading process* no *hinge once formed has been unloaded, the largest displacement so predicted will be the correct one.'*

i.e highest external work ⇒ smallest load factor (unsafe theorem)

Table 6.1

Last hinge	δ	θ_{AB}	θ_{BA}	θ_{BC}	θ_{CB}	Deflected shape
A or C	0	0	$-\frac{wL^3}{32EI}$	$+\frac{wL^3}{32EI}$	0	
B	$\frac{wL^4}{64EI}$	$+\frac{wL^3}{32EI}$	0	0	$-\frac{wL^3}{32EI}$	

Since the calculations are based on conditions at collapse there can be no indication of any hinge which may have formed and then disappeared. However, it is unusual for that to happen. Thus the largest deflection is normally correct.

6.2.3 Portal Frame Example

This second example brings out two more important points. The analysis is summarised in figure 6.5.

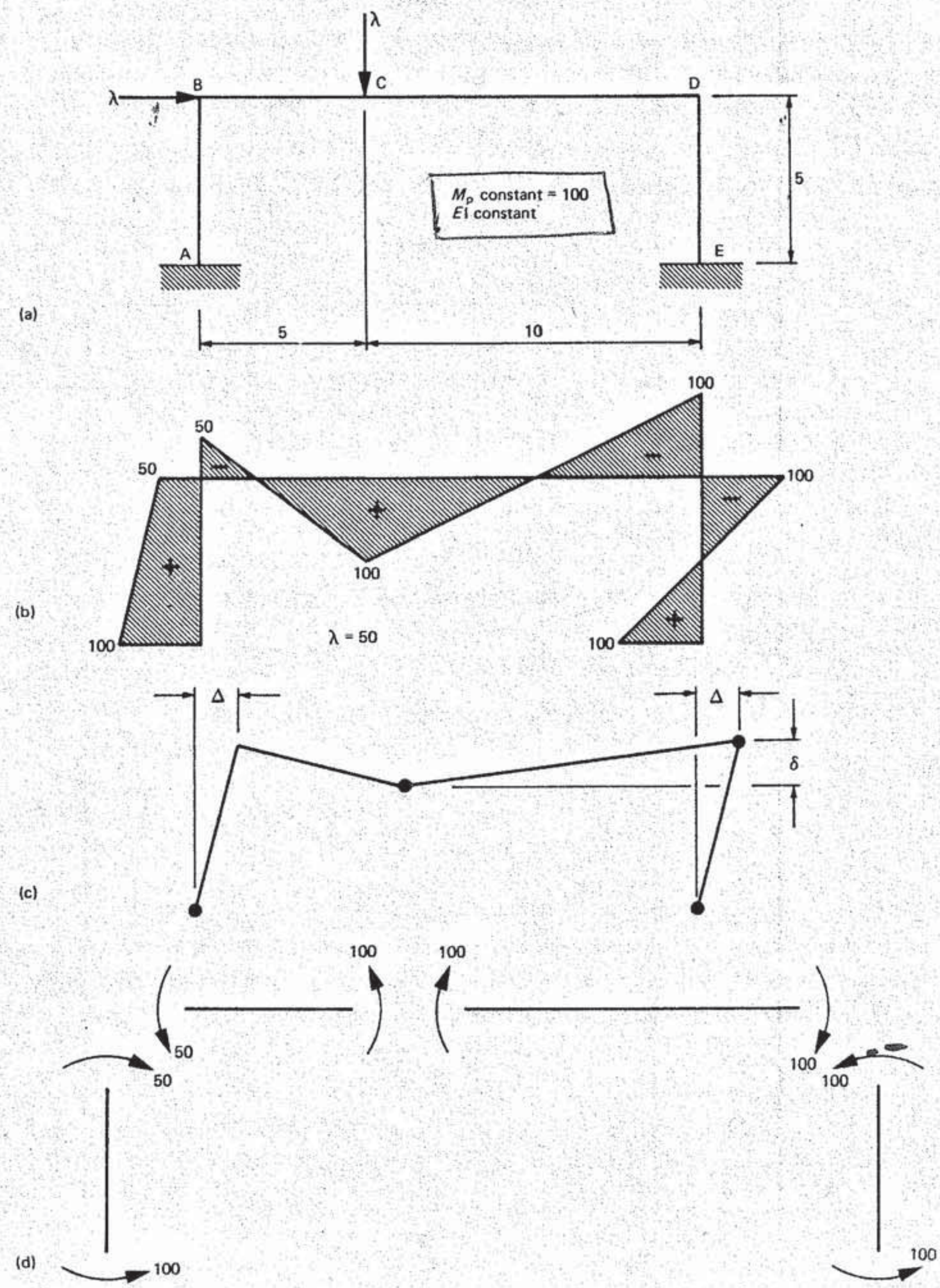


Figure 6.5

First Stage

The virtual work method shows that the frame collapses by the combined mechanism when $\lambda_c = 50$. The BMD at collapse and the collapse mechanism are shown in figures 6.5b and c.

Second Stage

The structure can be broken down into four members as in figure 6.5d. There are no fixed end moments in this case because the point loads are at the ends of members. There is no problem in deciding the direction of the end moments except at B. Here there is no plastic hinge in the mechanism. To decide on the direction of the end moments, imagine that the BM at B is increased until a hinge forms. The end moments act to resist the rotation of that imaginary hinge, as shown in figure 6.6.

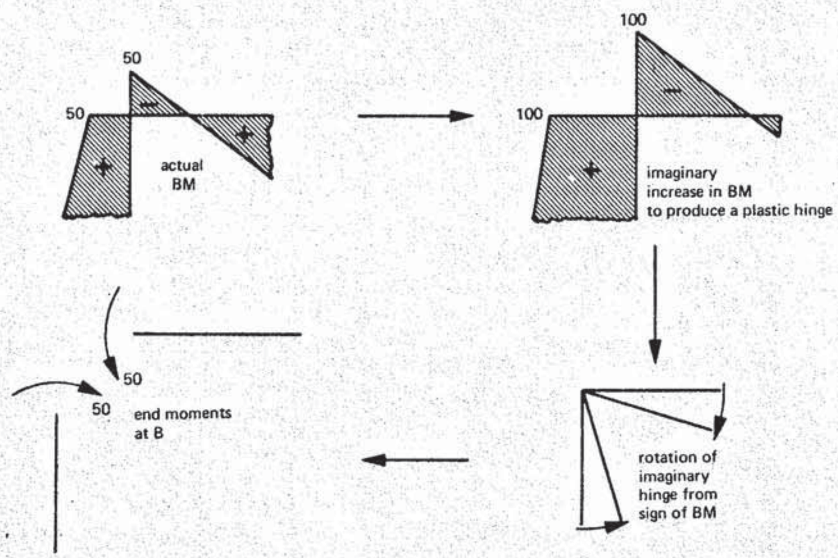


Figure 6.6

Two deflections are required in order to write down the slope deflection equations. It is assumed that the vertical deflection in the beam, δ, is small so that the tops of both columns move Δ horizontally. (This is the same assumption as used in calculating collapse loads.) The slope deflection equations are

$$\theta_{AB} = \frac{\Delta}{5} + \frac{5}{6EI}(-200 - 50) = \frac{\Delta}{5} - \frac{1250}{6EI}$$

$$\theta_{BA} = \frac{\Delta}{5} + \frac{5}{6EI}(+100 + 100) = \frac{\Delta}{5} + \frac{1000}{6EI}$$

$$\theta_{BC} = \frac{\delta}{5} + \frac{5}{6EI}(-100 + 100) = \frac{\delta}{5}$$